AN INTRODUCTION TO PROJECTIVE GEOMETRY

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PREFACE

This book is intended as an introductory account for senior-college and beginning graduate students—for the prospective teacher who is seeking a proper orientation of elementary mathematics, as well as the university student who lacks the preparation for an intelligent reading of the general treatises on higher geometry and the modern books on higher algebra. It represents the evolution of a course of lectures which I have given to this class of students for several years.

The adoption of the analytic method makes it possible to capitalize the student's collegiate training in algebra, analytic geometry and calculus, and at the same time to articulate the subject with his future mathematical work. Constant appeal is however made to geometric reasoning, the geometry not infrequently anticipating or suggesting the algebra. Indeed, the beautiful synthetic treatment of the conic, including Pascal's theorem and consequences, has been retained, followed by a full discussion of the conic both as a rational curve and as a ternary form.

Among the features of the book may be mentioned:

1. The abundance of exercises of a wide range of difficulty—from simple applications of the text to theorems which might have been included in the text itself.

2. Chapter I, which contains a systematic exposition of several topics, a knowledge of which is customarily assumed in the literature to the bewilderment of the beginner.

3. The principle of duality is introduced early and used constantly.

4. The algebraic as well as the geometric implications of infinite elements are considered.

5. Quadratic involution is developed from the projective standpoint, thus avoiding the metrical methods which are such a conspicuous flaw of most of the English texts.

6. The elements of binary forms and algebraic invariants are presented in a concrete and tangible fashion, intelligible to the class of students for which the book is designed. Practically the whole subject, including apolarity, is made to depend on a single simple process—the polar process.

7. The notion of group is introduced in connection with collineations and the development is carried out in some detail for the cyclic and dihedral groups in the binary and ternary domains.

8. Particular stress has been placed on the projective geometry of one dimension— the geometric interpretation of binary forms on the line, the conic and the rational cubic.

9. A chapter on non-Euclidean geometry exhibits the relationship of the standard metric geometries to one another as well as to projective geometry.

The arrangement permits considerable variation in the order of topics to accommodate the needs of different classes. For example, Chapter X (either part) might follow Chapter V, VI, or VII, or again Chapter XI, XII, or XIII might be taken up after Chapter VII.

The material will be found adequate for a year course. But the first six or seven chapters, which cover the classical theory, are sufficiently complete in themselves for a brief course. In this first part the connection between projective and Euclidean metric geometry is emphasized but the metric is everywhere subordinated to the projective. Chapters VIII–XII (approximately half the book), in which less attention is given to metrical specializations, are addressed to more mature readers. It is hoped that the young

graduate student will find these later chapters a welcome and readable introduction to invariant theory, rational curves and collineation groups.

I have ventured to introduce a new symbol ($\mathcal{L}$) for the line at infinity—a symbol which is distinctive, easy to make and suggestive of its own meaning.

I am much indebted to the standard treatises of Salmon and Clebsch and in the synthetic treatment of the conic in Chapter VI to Cremona's Projective Geometry, while the chapter on non-Euclidean geometry follows chiefly the exposition of Klein. With pleasure and gratitude I acknowledge also my deep obligations to my teachers, Professors Morley, Coble and Hulburt. Finally to D. C. Heath and Company for their uniform courtesy, and for their scrupulous care with the printing and engraving, I wish to express my very great appreciation.

R. M. Winger.

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CHAPTER I

ESSENTIAL CONSTANTS

1. Parameters.—The student is already familiar with two distinct conceptions of the literal constants in an equation. Thus we commonly say that the equation

$$x^2 + y^2 = k, \tag{1}$$

where x and y are rectangular coördinates, represents a circle with center at the origin and radius equal to $\sqrt{k}$. That is, we think of k as having a particular or known value, though arbitrarily chosen, and our discussion is limited to a single circle. On the other hand, we may suppose k to run through a succession of values when (1) represents not one circle but a *family* of circles, namely all circles with center at the origin. Under the latter interpretation k is called an *arbitrary constant* or *parameter*.[1] For every choice of k there is a definite circle and there are an infinite number of choices, *viz.*, all values from $-\infty$ to $+\infty$. But since there is a single arbitrary constant in its equation, this is called a *singly infinite* or *one-parameter* family of circles. Or we may say that the family contains ∞^1 members.

[1] The student should distinguish between the variation of k and that of x and y; for k varies *arbitrarily* while x and y must satisfy relation (1). Moreover for any particular circle of the system the value of k is fixed while x and y range through an infinity of values.

The general equation of a line,

$$ax + by + c = 0, \tag{2}$$

appears to contain three arbitrary constants, a, b, and c. However they are not independent for the equation may be divided by any one of them reducing the number to two, thus

$$x + \frac{b}{a}y + \frac{c}{a} = 0, \quad \text{or} \quad x + py + q = 0. \tag{3}$$

This fact is expressed by saying that the general equation of a line contains two *essential constants.* Since there are infinitely many choices for p and also for q, it is customary to say that there are a *double infinity* (∞^2) equations of the form (2), or that there are ∞^2 lines in the plane;[1] or again that the totality of lines in the plane constitutes a two-parameter family.

Definitions.—Any condition which involves a linear relation among the coefficients of an equation is called a *linear condition* on the equation. *Independent* linear conditions are those which give rise to independent linear equations, *e. g.* (4) and (5) below. Similar definitions apply to quadratic conditions and the higher cases.[2] A *linear family* or *system* is one in which all the parameters enter to the first degree. In a quadratic system at least one of the parameters is of the second degree, and so on. A set of equations or curves are said to be *linearly independent* when they do not belong to a linear system.

2. The geometrical significance of the constants in an equation will now be illustrated. Suppose that the line,

[1] The student is warned not to confuse the present notation with that for exponents. It is only a convenience of language derived by analogy. For if there were n choices for p and n for q there would be n^2 choices for p and q combined.

[2] In an equation such as $x^2/a^2 + y^2/b^2 = 1$, the coefficients are to be considered as $1/a^2$, $1/b^2$ and 1 when the degree of a condition is in question. Thus $1/a^2 + 1/b^2 = 1$ is a linear condition though of the fourth degree in a and b.

§1, is restricted to pass through a fixed point as for example (1, 1). The coördinates of the point must satisfy the equation (2), hence

$$a + b + c = 0 \quad \text{or} \quad 1 + p + q = 0. \tag{4}$$

Now only one of the coefficients say p can be chosen at random, the second, q, being determined by relation (4). The number of essential constants is reduced to one,—geometrically there is a single infinity of lines through a point, or the lines through a point make up a one-parameter family. If the line is required to pass through a second point (2, 3) we must have

$$2a + 3b + c = 0, \quad \text{or} \quad 2 + 3p + q = 0. \tag{5}$$

Equations (4) and (5) are sufficient to determine the values of p and q and the ratios of a, b and c, thus $a : b : c = 2 : -1 : -1$ and a definite line is thereby obtained

$$2x - y - 1 = 0.$$

Again the general equation of the second degree in two variables

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{6}$$

contains six homogeneous, five essential constants. Geometrically, there are ∞^5 conics in the plane, or the aggregate of conics in the plane composes a five-parameter family. If equation (6) represents a parabola

$$h^2 - ab = 0. \tag{7}$$

Although (7) is not a linear condition we still say that the number of essential constants is thereby reduced to four.[1] If we impose the condition that the axis of the parabola be parallel to the y-axis, the equation can be written in the form

$$y = a_1x^2 + b_1x + c_1 \tag{8}$$

[1] Since if a and b are given there are only two choices for h whereas an arbitrary constant may assume an infinity of values.

which contains three essential constants. If we ask further that the focus lie on the y-axis, (8) takes the form

$$y = a_1x^2 + c_1. \tag{9}$$

If in addition the vertex be at the origin, the equation becomes

$$y = a_1x^2. \tag{10}$$

Finally if (10) pass through the point (1, 1),

$$y = x^2. \tag{11}$$

Thus each condition imposed on the curve diminishes the number of essential constants by one. Or we may say: Of the ∞^5 conics in the plane, ∞^4 are parabolas. There are ∞^3 parabolas which have their axes parallel to the axis of y; of these, ∞^2 have foci on the y-axis; of this family there are ∞^1 each with vertex at the origin. Finally there are ∞^0 parabolas (in this case there is a unique curve) satisfying all the above conditions and passing through the point (1, 1).

Obviously a single statement in English may involve two or more mathematical conditions. For example, it is two conditions for a parabola to have its vertex at a given point (as the origin) or its axis coincident with a given line (as the y-axis). However if the point is on the line either of these followed by the other makes up a total of only three.

3. The meaning of non-linear conditions will now be examined briefly. If the conic (6) be made to pass through four arbitrary points, four independent linear equations connecting the five essential constants will be obtained. By means of these equations four of the constants can be expressed linearly in terms of the fifth and we have a one-parameter family of conics. To find the parabolas of the family we must combine equation (7), a quadratic, with the other four. Thus since four linear equations and a quad-

ratic among five unknowns have in general two common solutions, we shall find usually two parabolas through the four points.

As a second illustration let us seek the equation of a line which passes through the origin and is tangent to the circle

$$x^2 + y^2 - 2x - 6y + 8 = 0. \tag{12}$$

The equation of the line may be taken as

$$y = ax + b. \tag{13}$$

This will pass through the origin if $b = 0$, a linear condition. The abscissas of the points of intersection of line and circle are given by the equation

$$(1 + a^2)x^2 - 2(1 + 3a)x + 8 = 0. \tag{14}$$

The line will be tangent to the circle if the roots of this equation are equal, *i. e.*, if

$$(1 + 3a)^2 - 8(1 + a^2) = 0, \tag{15}$$

a quadratic condition. We find thus two solutions to the problem, *viz.*: $b = 0$; $a = 1, -7$.

We see then that while a quadratic condition has the same effect as a linear in reducing the number of constants in an equation, its presence in a set of conditions destroys the uniqueness of the solution. For example in the preceding problem the two constants in (13) were subject to two conditions which are sufficient to determine them. Since however one condition is linear and one quadratic, we find not one line but two fulfilling the requirements. Many other illustrations of this sort will occur to the student.

4. The results of the preceding paragraphs may now be generalized and summarized as follows:[1]

[1] In all these theorems we have supposed that no two of the equations concerned are inconsistent, *i. e.*, that no two of the conditions are contradictory. That would happen for example if we ask that the circle, $x^2 + y^2 + ax + by + c = 0$, pass through the points (0, 0), (1, 0), (2, 0). We should have $c = 0$, $1 + a = 0$, $2 + a = 0$. We shall see, however, that for the purposes of projective geometry there are no inconsistent equations.

1°. *n homogeneous constants are equivalent to $n - 1$ essential constants.* An equation with n essential constants represents an n-parameter system or family which contains ∞^n members.

2°. *If n essential constants are connected by r independent relations, linear or non-linear, the number is reduced to* $n - r$; for each relation enables us to express one of the constants in terms of the others. We can thus eliminate in succession r of the constants.

3°. *A linear system with n essential constants can be made to satisfy in one and only one way n independent linear conditions.* For the conditions give rise to a system of n independent non-homogeneous linear equations which possess a unique solution.

4°. *Similarly an equation with n essential constants can be subjected to n independent conditions, linear or non-linear, giving rise to a finite number of solutions, designated by ∞^0.* In particular a curve may be made to pass through as many arbitrary points as there are essential constants in its equation, though the solution is not necessarily unique; and the curve may degenerate.

5. If each of the homogeneous constants in an equation is equal to zero, the equation is said to *vanish identically.* Obviously it is then satisfied by arbitrary values of the variables. Conversely if an equation is true for all values of the variables the coefficients are separately equal to zero. Indeed if an equation with n essential constants is satisfied by $n + 1$ independent sets of values of the variables it must vanish identically. For we should have $n + 1$ independent relations among n non-homogeneous coefficients whereas they can sustain but n, unless each is zero. Hence

5°. *It is $n + 1$ conditions for an equation with n essential constants to vanish identically.*

Similarly it is $n + 1$ conditions for a polynomial with n

essential constants to be identical with another of the same degree for corresponding coefficients must be equal, *i. e.*, the equation formed by transposing all the terms to one side must vanish identically.

EXERCISES

1. Show that the totality of circles in the plane is ∞^3. How many circles are there

(*a*) with given center
(*b*) with given radius
(*c*) through the origin
(*d*) tangent to a fixed line
(*e*) tangent to a fixed line at a given point
(*f*) through a point and tangent to a line?

Write the equation of each family here mentioned.

2. Write the equation of all lines

(*a*) through the origin
(*b*) through the point $(2, -1)$
(*c*) parallel to the line $3x - 4y + 5 = 0$
(*d*) perpendicular to the same
(*e*) at a distance r from the origin.

What sort of a family does each represent? From (*e*) infer the number of tangents to a circle.

3. Find the equation of all circles

(*a*) through the point $(1, 3)$
(*b*) through $(1, 3)$ and $(-2, 5)$
(*c*) with center on the line $x + 2y - 3 = 0$.

How many circles in each system?

4. How many points are there

(*a*) in the plane
(*b*) on a given line
(*c*) on any curve?

5. What kind of condition is it for a circle to touch a given line? Show why more than one circle can be drawn to touch three given lines.

6. Enumerate the quadratic equations in one variable. How many of these have

(*a*) equal roots
(*b*) one root zero
(*c*) a fixed number for the sum of the roots
(*d*) the sum of the roots equal to the product?

7. How many essential constants in the general equation of degree n in one variable?

8. Find the equations of the parabolas through (0, 0), $(-2, 2)$, $(-\frac{3}{2}, 1)$, (0, 4). *Ans.* $y^2 - 2x - 4y = 0$, $16x^2 + 8xy + y^2 + 14x - 4y = 0$.

9. What geometrical ideas are in conflict in the footnote, § 4?

10. A system of k linear, non-homogeneous independent equations in n variables, $k \leqq n$, have how many solutions? If $k > n$ how many conditions to have a solution? State corresponding theorems for k homogeneous equations in $n + 1$ variables. Do similar theorems hold for non-linear equations?

6. Necessary and sufficient conditions.—If a relation inevitably follows a certain hypothesis or event, that relation is called a *necessary* condition for the event. On the other hand if an event is an unavoidable consequence of a relation, the relation is called a *sufficient* condition for the event. A condition may be (a) necessary, (b) sufficient or (c) both necessary and sufficient. To show that a condition is necessary and sufficient entails the proof of a proposition and its converse.

Thus $h^2 - ab = 0$ is a necessary condition for equation (6) §2 to represent a parabola. For if the conic be a parabola $h^2 - ab$ is necessarily zero. It is however not sufficient since it may be fulfilled without the conic being a parabola, *viz.*, if the equation represent two parallel or two coincident lines. But we may say that the necessary and sufficient condition for a *proper* conic to be a parabola is $h^2 - ab = 0$.

Again a necessary but not sufficient condition for a point of inflexion is that the second derivative vanish at the point. Likewise a sufficient but not necessary condition for an inflexion is that the second derivative vanish and the third derivative be different from zero.[1]

[1] See for example Osgood, *Calculus*, revised edition, p. 55.

EXERCISES

1. Show that the necessary and sufficient condition that the quadratic equation $ax^2 + bx + c = 0$ have equal roots is $b^2 - 4ac = 0$.

2. Find the necessary condition for the quadratic to have one root zero. Show that the condition is also sufficient.

3. Show that the necessary and sufficient condition for an algebraic curve to pass through the origin is that the constant term vanish.

4. Find necessary conditions that the general equation of the second degree in two variables represent a circle. Are these conditions sufficient?

7. We shall now enumerate the constants in the general equation of degree n in two non-homogeneous variables. Arranged in ascending powers, the terms of such an equation are of the following types:

$$
\begin{array}{ccccccccccc}
 & & & & & a & & & & & \\
 & & & & bx & & cy & & & & \\
 & & & dx^2 & & exy & & fy^2 & & & \\
 & & gx^3 & & hx^2y & & ixy^2 & & jy^3 & & \\
 & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
k_0x^n & & k_1x^{n-1}y & & k_2x^{n-2}y^2 & \ldots & k_{n-1}xy^{n-1} & & & & k_ny^n.
\end{array}
$$

The number of constants thus forms an arithmetic progression of $n + 1$ terms with first term 1, last term $n + 1$ and common difference 1. The sum of the progression, *i. e.*, the number of homogeneous constants is $\dfrac{n+1}{2}(n + 2)$. The number of essential constants therefore is

$$\frac{n^2 + 3n + 2}{2} - 1 = \frac{n}{2}(n + 3).$$

A knowledge of the essential constants explicitly or implicitly contained in any system of equations of geometric configuration is of primary importance as will be abundantly verified in the sequel. A mere counting of the constants would appear a simple matter and so it is if the

equation is general. But it becomes more difficult when several equations or figures, restricted by various relations, are considered. Always great care must be exercised for many subtleties will be encountered. We have seen that the number of constants limits the *number* of conditions that may be imposed but the *nature* of the conditions must be taken into account as well. Thus a general plane cubic curve has nine constants and it is one condition for a curve to have a double point but it must not be argued thence that a cubic can have nine double points.

EXERCISES

1. Find the number of essential constants in the general equation in two non-homogeneous variables of degree 3, 4, 5, 6, obtaining thus the number of arbitrary points necessary to determine the curves of those degrees.

2. Find the number of essential constants in the general equation of degree n in three non-homogeneous variables.

3. Show that the number of homogeneous constants in an equation of degree n homogeneous in k variables is

$$\frac{(n+1)(n+2)\ .\ .\ .\ (n+k-1)}{1.2.3\ .\ .\ .\ (k-1)}$$

8. Before leaving this subject it will be instructive to examine the modern conception of dimension, a term imperfectly understood by most young students. The number of essential constants (parameters) in a family is called its *dimension.* In this connection the word family is frequently replaced by *manifold.* Thus an n-parameter family and an n-dimensional manifold are equivalent. Since the points of a line belong to a one-parameter family, the line is sometimes called a one-dimensional manifold of points or a one-dimensional geometric *form.* But equally well the points of a curve belong to a one-parameter family (Ex. 4 §5) so that a curve is also a one-dimensional form.

Analytically, the line consists of all points whose coördinates satisfy the equation $ax + by + c = 0$ while a (plane) curve is the manifold of points whose coördinates satisfy an equation $f(x, y) = 0$. The difference between the two cases is now obvious,—in the first the parameters[1] are connected by a linear relation, in the second the relation is of higher degree. In space a curve is defined by two equations $f_1(x, y, z) = 0$, $f_2(x, y, z) = 0$, which are simultaneously satisfied. Here again at least one of the equations is of higher degree than the first. The line then is merely the simplest one-dimensional manifold of points.

Similarly a surface is a two-dimensional manifold of points for its equation, $f(x, y, z) = 0$, contains three parameters subject to one condition. In particular a plane is the simplest two-dimensional manifold of points since its equation is linear. But the dimension of any given manifold, as a plane, depends upon the element by which it is generated. A plane is two-dimensional in lines as well as in points for the equation of an arbitrary line contains two essential constants. But as the plane contains ∞^3 circles and ∞^5 conics it is a three-dimensional manifold of circles and a five-dimensional manifold of conics.

9. A distinction is usually made between manifolds which are linear and those which are not, when a linear manifold of n dimensions is termed a *space*[2] of n dimensions and designated by S_n[3]. More especially a space is restricted to mean a space of points. In a still more restricted sense, space refers to the ordinary three-dimensional space.

The point, line and plane are thus spaces of zero, one

[1] Here x and y are the parameters while the coefficients are definite constants.

[2] Space is sometimes used synonymously with manifold, *linear space* or *flat space* denoting space as defined above.

[3] Similarly a manifold of n dimensions will be denoted by M_n. The manifold is said to have *n degrees of freedom.*

and two dimensions respectively. Analytically they are defined by the equations

$$
\begin{aligned}
S_0&: \quad a_0x + b_0 = 0,\\
S_1&: \quad a_1x + b_1y + c_1 = 0,\\
S_2&: \quad a_2x + b_2y + c_2z + d_2 = 0.
\end{aligned}
$$

Each equation represents a space of a given dimension lying in a space of the next higher dimension. So far no one will object for we have not gone beyond ordinary space which is accepted by all.

But why should we stop? The algebra certainly continues and there is no logical reason why the geometry should not follow. It is true that we ordinarily begin the study of analytic geometry by setting up a geometrical frame-work,—the coördinate axes,—adopt the point as element and introduce the coördinates as numbers associated with the point. This practice may be a pedagogical advantage but it is not a logical necessity. We might begin with the coördinates and introduce the point as a geometrical *interpretation* of the numbers. We could thus postulate the point, line and plane as geometrical images of the equations above. We should then quite naturally say that

$$S_3: \quad a_3x + b_3y + c_3z + d_3t + e_3 = 0$$

represents a space of three dimensions lying in a space S_4 of four dimensions. Generally, the equation

$$S_{n-1}: \quad a_0x_0 + a_1x_1 + a_2x_2 + \ . \ . \ . + a_{n-1}x_{n-1} + a_n = 0,$$

containing n numbers x connected by a linear relation, would define an ∞^{n-1} of points, *i. e.*, a space of $n - 1$ dimensions, lying in a space of n dimensions. We could thence define the dimension of a space as the number of coördinates necessary to fix a point (element) in it.

While we may have manifolds of n dimensions entirely confined to a space of lower dimensions, it may require an n-dimensional space to contain a manifold of lower dimensions. There are, *e. g.*, curves of the nth degree whose points cannot be embraced in a space of less than n dimensions.[1] A curve of the nth degree however must be completely contained in a space of n, or lower dimensions.

Likewise a point space S_k which contains enough points of an S_{k+1} to determine it lies wholly in the S_{k+1}.

The timid may console themselves with the reflection that the geometry of four or higher dimensions is, if not a necessity, certainly a convenience of language,—a translation of the algebra,—and let the philosophers ponder the metaphysical questions involved in the idea of a point space of higher dimensions. But no one with a clear conception of the nature of mathematics will hesitate to admit the geometry of hyperspace[2] into the society of mathematical disciplines on a basis of complete equality. It should not be forgotten that we actually have linear spaces of n dimensions existing in the plane and in ordinary space. The abstract theory of a point space of five dimensions for example is identical with that of the system of conics in the plane. The properties of one imply all the properties of the other and in a mathematical sense one is just as real and existent as the other.

EXERCISES

1. Show that two spaces S_3 of an S_4 have a linear manifold M_2 in common, *i. e.*, intersect in a plane. Generally, two S_k's of an S_{k+1} intersect in an S_{k-1}.

2. Show that two planes in four dimensions meet in a point; that two planes in five dimensions do not meet.

[1] Such curves are called *norm* curves and are always *rational.*

[2] Hyperspace denotes a space of more than three dimensions though sometimes in a narrower sense refers to S_4.

3. Prove that mS_{n-1}'s, in S_n, $m \leq n$, meet in an S_{n-m}. Thence or otherwise show that an S_i and an S_k in S_n intersect in an S_{i+k-n}, $i + k \geqq n$.

4. State analytic reasons why two independent points determine a line and three a plane. In a similar manner show that four points in general determine an S_3. And generally, $n + 1$ points determine a space of n dimensions.

5. What is the dimensionality of ordinary space S_3 when the element is (*a*) the sphere, (*b*) the general quadric surface, (*c*) the line, (*d*) the circle?

CHAPTER II

DUALITY

No demonstration can be held valid in *method*, or as touching the essence of the subject matter, in which the indifference of the duadic law is departed from. Until these recent times, the analytic method of geometry, as given by Descartes, had been suffered to go on halting as it were on one foot. To Plücker was reserved the honor of setting it firmly on its two equal supports by supplying the complementary system of coördinates.—SYLVESTER.

10. Plücker line coördinates.—We have seen that a line in the plane possesses two characteristic constants, *viz.*, the essential constants in the general equation. These may be chosen in various ways. For example any two of the six ratios of the coefficients in the equation $ax + by + c = 0$ might be selected.[1] Geometrically, to choose the constants simply amounts to choosing the geometrical conditions that fix the position of the line in the plane. Thus when the equation is written in the form

$$y = tx + y_0$$

the constants are the slope t and the y-intercept y_0. When the Hessian normal form

$$x \cos \alpha + y \sin \alpha = p$$

is used the constants are the length p of the perpendicular upon the line from the origin and the angle α which that perpendicular makes with the x-axis.

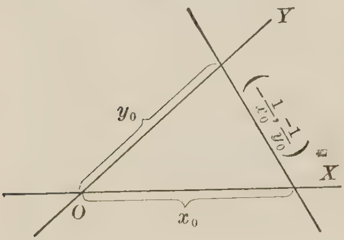
Since the two constants however chosen are sufficient to locate the line in the plane they may be called *coördinates* of

[1] Except of course the pairs like a/b and b/a.

the line or *line coördinates.* When the equation of a line is reduced to the form

$$ux + vy + 1 = 0, \tag{1}$$

where x and y are either rectangular or oblique point coördinates, the numbers u and v are called the *Plücker line coördinates.* Geometrically *these numbers represent the negative reciprocals of the intercepts of the line* as may be verified by setting x and y successively equal to zero.



If the equation of a line is given the Plücker line coördinates, or the line coördinates as we shall call them henceforth, can be found by reducing the equation to the form (1). Conversely if the coördinates of a line are known its equation can be written at once. Thus the coördinates of $4x + 5y + 2 = 0$ are $(2, \frac{5}{2})$; while the equation of the line (1, 3) is $x + 3y + 1 = 0$.

In the preceding we have tacitly supposed that u and v were definite constants. Let us now examine their meaning when considered as variables. If we ask that line (1) pass through the point (2, 3) we must have

$$2u + 3v + 1 = 0. \tag{2}$$

For every pair of values of u and v which satisfy (2) we have a line but every such line contains the given point. Thus when $u = 1, v = -1$ and (1) becomes $x - y + 1 = 0$ which obviously passes through (2, 3).

We may say then that (2) is the relation which is satisfied by the coördinates of all the lines through the point (2, 3); in other words (2) *is the equation of the point* (2, 3) *in line coördinates.* If the coördinates of a point are known its

equation can be written down at once. Moreover if the equation of the point is given its coördinates can be found by a method analogous to that for a line. Thus the general linear equation in u and v

$$au + bv + c = 0 \tag{3}$$

represents a point whose coördinates are $(a/c, b/c)$.

But how shall we regard an equation like (1) which is linear in both point and line coördinates? The answer to the question will be best understood perhaps by summarizing the discussion in parallel columns

(*a*) If u and v are fixed constants and x and y are variables

$$ux + vy + 1 = 0$$

is the equation of the line (u, v).

(*b*) If u and v are parameters the equation gives a doubly infinite system of lines, *viz.*, all lines in the plane.

(*a'*) If x and y are fixed constants and u and v variables

$$ux + vy + 1 = 0$$

is the equation of the point (x, y).

(*b*) If x and y are parameters the equation gives a doubly infinite system of points, *viz.*, all points in the plane.

Since in either case the equation is the condition that the point (x, y) lie on the line (u, v), (1) is frequently called the *incidence relation* or the *equation of united position of point and line.*

Another example will make still clearer the significance of line coördinates.

(*c*) If the points (x_1, y_1) and (x_2, y_2) lie on the line

$$ax + by + c = 0$$

(*c'*) If the lines (u_1, v_1) and (u_2, v_2) pass through the point

$$au + bv + c = 0$$

we must have

$$ax_1 + by_1 + c = 0$$
$$ax_2 + by_2 + c = 0.$$

The condition that these three equations be consistent, *viz.*,

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0,$$

is the equation of the line joining the two points.

we must have

$$au_1 + bv_1 + c = 0$$
$$au_2 + bv_2 + c = 0.$$

The condition that these three equations be consistent, *viz.*,

$$\begin{vmatrix} u & v & 1 \\ u_1 & v_1 & 1 \\ u_2 & v_2 & 1 \end{vmatrix} = 0,$$

is the equation of the point of intersection of the two lines.

(*d*) Suppose now that the two points are given not by their coördinates but by their equations, thus

$$a_1u + b_1v + c_1 = 0$$
$$a_2u + b_2v + c_2 = 0.$$

The equation of the line joining these points is found to be

$$\begin{vmatrix} x & y & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

or $|bc|x + |ca|y + |ab| = 0.$

Cor. If a third point

$$a_3u + b_3v + c_3 = 0$$

lie on the line

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

(*d′*) Similarly, the equation of the point of intersection of the lines

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

is

$$\begin{vmatrix} u & v & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

or $|bc|u + |ca|v + |ab| = 0.$[1]

Cor. If a third line

$$a_3x + b_3y + c_3 = 0$$

pass through the point

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

[1] By $|ab|$ we mean the determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$. Similarly the last determinant of this article is abbreviated to $|abc|$.

EXERCISES

1. Find the coördinates of the lines (*a*) $4x - 5y = 7$, (*b*) $x + 3 = 0$, (*c*) $y = mx + b$, (*d*) $x \cos \alpha + y \sin \alpha = p$, (*e*) $ax + by + c = 0$, (*f*) $x/a + y/b = 1$, (*g*) $yy_1 = p(x + x_1)$, (*h*) $x_1y + y_1x = 2c$.

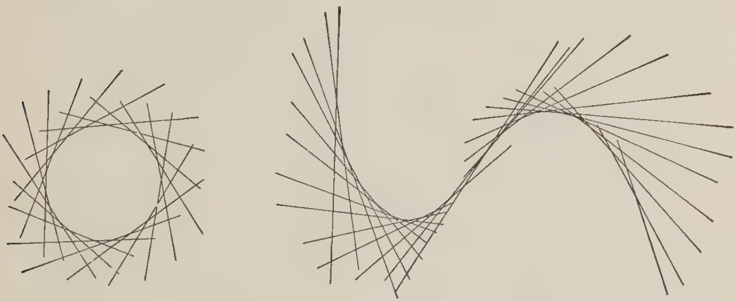
2. Plot the lines $(1, 1)$, $(\frac{1}{3}, \frac{2}{5})$, $(0, a)$, $(b, 0)$ and write their equations.

3. Write the equations of the points $(3, 5)$, $(\frac{3}{2}, -7)$, $(a, 0)$.

4. Find the coördinates of the points and lines in (*c*) and (*d*) above.

5. Write the equation of the point of intersection of $x - 3y - 2 = 0$ and $5x + 4y + 6 = 0$.

11. Line locus.—The foregoing suggests for the line a rôle analogous to that played by the point in the analytic geometry of Descartes. There a "curve" is regarded as the locus of a point and a relation connecting a pair of variables is the "equation of such a curve" or *point locus* only in virtue of the interpretation of the variables as point coördinates. The whole of (plane) Cartesian geometry is but a translation of the theory of algebraic equations in two variables into the language of geometry. In this connection we think of a point as an *element*, *i. e.*, not divided into parts. While a line is an assemblage of points—a point locus.



Equally well a line (of indefinite extent) may be considered as having an existence quite apart from any of the points which lie on it,—in other words a line may be considered an element. The notion will be clear if one con-

ceives a line not as traced by a pencil point but as stamped by a straight edge. Let us fix our attention on the line as an element, having introduced the notion of line coördinates. Suppose now that the line moves subject to some condition, *e. g.*, that it shall always be tangent to a circle. The moving line will coincide in succession with all the tangents to the circle. The aggregate of these lines constitute a *line locus.* Similarly the system of tangents to any point locus is a line locus. Thus a line locus is a one-parameter family of lines just as a point locus is a one-parameter family of points. More precisely we may say

A point locus is the assemblage of all points whose coördinates satisfy an equation of the form $f(x, y) = 0$.

The degree of this equation is called the *order* of the point locus.

The points of a point locus of the first degree lie on a line, or a point locus of the first order is a line.

The junctions of consecutive points of the point locus are lines of a line locus.

A line locus is the assemblage of all lines whose coördinates satisfy an equation of the form $\phi(u, v) = 0$.

The degree of this equation is called the *class* of the line locus.

The lines of a line locus of the first degree pass through a point, or a line locus of the first class is a point.

The intersections of consecutive lines of the line locus are points of a point locus.

12. Two notions of curve.—It is now evident that the term "curve" is ambiguous and we have to distinguish between *point curve* which is a locus of points and a *line curve* which is a locus of lines. We might of course introduce the concept curve as a thing in itself,—like the concept line considered as an element, Imagine for example a circle as stamped by a material ring. Such a "circle" might serve to carry an infinite number of points, as beads

on a string. Moreover an infinite number of lines could be drawn to touch the circle. The infinite system of points would constitute a point locus and the infinite system of lines a line locus.

More generally let us fix our attention upon a point and line in the united position both free to move, the point along the line and the line about the point. Suppose now that the point moves generating a point locus and that the line rotates about the point in such a way as to be always tangent to the point locus. Then the line will generate a line locus. The equation in x and y satisfied by the coördinates of the point would be the equation of the point locus and the equation in u and v satisfied by the coordinates of the line would be the equation of the line locus But the system either of points or lines would determine a curve in the sense here used. Hence the first equation might properly be called the *equation of the curve in point coördinates* and the second the *equation of the curve in line coördinates.*

We may then use point locus and line locus to denote the respective assemblages of elements and reserve curve for the thing associated with both. *If, however, we wish to consider a curve as made up of elements it is only fair to regard it as both a locus of points and a locus of lines.*

13. The principle of duality.—We have seen (§10) that an equation of the first degree in two variables may represent either a line or a point. Similarly any equation, $f(\alpha, \beta) = 0$, between two variables will represent either a point locus or a line locus according as α, β are interpreted as point or line coördinates. Likewise any property of the equation will have a twofold geometric interpretation, one for the point locus and one for the line locus We are now in a position to enunciate for the plane one of the most far-reaching principles of mathematics known as the *principle*

of duality or reciprocity: The theory of algebraic equations in two variables gives rise to two geometries according as the variables are interpreted as point or line coördinates. These two geometries are called *dual* geometries, or if we choose to consider them as one geometry we say that the theorems of geometry occur in pairs. Corresponding ideas, figures, properties are known as *dual* or *reciprocal.*

Let us illustrate how single theorems of algebra can be translated into pairs of dual theorems of geometry. We shall give first the algebraic statement then the geometric theorems side by side immediately below. All the theorems are stated for two variables.

(1) Two linear equations have in general one common solution.

(1′) Two lines in general have one common point, their intersection.

(1″) Two points in general have one common line, their junction.

(2) If two linear equations have more than one common solution they have all their solutions in common.

(2′) If two lines have more than one point in common they coincide.

(2″) If two points have more than one line in common they coincide.

(3) A linear equation and an equation of degree n have precisely n solutions, real or imaginary.

(3′) A line and a point locus of order n have precisely n points, real or imaginary, in common, *i. e.*, the order of a curve is the number of points in which it cuts a line.

(3″) A point and a line locus of class n have precisely n lines, real or imaginary, in common, *i. e.*, the class of a curve is the number of tangents which can be drawn from a point.

(4) Two equations of degrees m and n have mn common solutions.

(4′) Two curves of orders m and n have mn common points.

(4″) Two curves of classes m and n have mn common lines.

In the preceding we have assumed the truth of the algebraic statement. We derived thence the geometric theorems by supposing that the variables were point coördinates on the one hand and line coördinates on the other. *But since the algebra of dual theorems is identical, the proof of either establishes the validity of both.* Therein lies the great power of the principle of duality. Its beauty consists in the symmetry with which it invests geometry.

The procedure in any instance is now clear. A theorem is proved either for point loci or line loci and the reciprocal theorem is merely stated, its proof being implied in the other.[1] Moreover an inspection of the illustrations above will indicate that the statements of dual theorems differ essentially only in certain technical terms. Naturally each concept in the geometry of point loci has a dual concept in the geometry of line loci so that a theorem in either is carried over into the other simply by the interchange of dual expressions.

14. To facilitate the process we collect into a vocabulary the dual terms already used.[2] The list will be supplemented as new words are introduced.

[1] Obviously, having established the principle of duality, the proof of a theorem need not be analytical and the algebra may be dispensed with altogether. We have simply used the algebraic means of approach.

[2] In space a point and plane are dual elements while the line is self dual. Similarly in S_4 a point (S_0) is the dual of a space (S_3) and a line is dual to the plane. There is no self-dual element. Generally in S_n an S_{i-1} and an S_{n-i} are dual spaces. Consequently there are self-dual spaces only when n is odd.

1. point	1. line
2. junction of two points	2. intersection of two lines
3. point locus	3. line locus
4. order	4. class
5. tangent (junction of consecutive points)	5. point of contact of tangent (intersection of consecutive lines)
6. curve (locus of points and lines)	6. curve (locus of lines and points)

When a point is on a curve, for brevity we shall also say that the curve is *on* the point.

15. Line equation derived from point equation.—Having chosen (§12) to regard a curve as equally a locus of points and a locus of lines, it is no longer quite appropriate to speak of *the* equation of the curve for there are *two* equations involved. Since however either defines the curve it should be possible to pass from one to the other. Suppose now we are given the relation in x and y which is satisfied by the points of the curve. The task is to find the relation in u and v satisfied by the lines (tangents) of the curve. In other words given the equation of the curve in point coördinates we are to seek the equation of the curve in line coördinates.

Example. Let the point equation be

$$x^2 + y^2 = r^2.$$

The line $ux + vy + 1 = 0$ will cut the circle in two points the abscissas of which are given by the quadratic

$$(u^2 + v^2)x^2 + 2ux + 1 - v^2r^2 = 0.$$

The roots of this equation will be equal and the points will coincide if

$$u^2 - (u^2 + v^2)(1 - v^2r^2) = 0,$$

or

$$u^2 + v^2 = 1/r^2.$$

Since this is an equation in u and v it represents a line locus. But it is the condition which must be satisfied by all lines which are tangent to the circle. It is therefore the locus of such lines or *the line equation of the circle.*

The student must be very careful to distinguish between the dual of a curve in point coördinates and its line equation. The dual of a point locus is found by replacing x and y in the equation by u and v,—*e. g.*, the dual of $f(x, y) = 0$ is $f(u, v) = 0$ and conversely. The equations are therefore of exactly the same form. The line equation on the other hand is not only of different form in general but, excepting in special cases, it is not even of the same degree.

EXERCISES

1. State the duals of the following:

(*a*) If the vertices of a triangle are points of a curve the triangle is said to be inscribed in the curve.

(*b*) If the coördinates of a point satisfy the point equation of a curve the point lies on the curve.

(*c*) The coördinates of the common points of two curves are the simultaneous solutions of the point equations of the curves.

(*d*) The line equation of a curve, $f(x, y) = 0$, is the condition, $\phi(u, v) = 0$, that an arbitrary line (u, v) shall be tangent to the curve, *i. e.*, have two consecutive points in common with the curve.

(*e*) It is a linear condition for a point curve to pass through a point. Hence a general curve of order n is uniquely determined by $\frac{1}{2}n(n + 3)$ arbitrary points.

(*f*) It is a quadratic condition for a point conic to touch a line. See Ex. 5 below.

2. Through four points and tangent to a line how many conics can be drawn? Dualize.

3. Find the line equation of (*a*) the parabola $y^2 = 4ax$; (*b*) the ellipse or hyperbola $x^2/a^2 \pm y^2/b^2 = 1$. Write the duals of each.

4. Show that the line equation of the hyperbola $4xy = k$ is $uv = 1/k$. Thence prove that the area of the triangle formed by a tangent to an equilateral hyperbola and its asymptotes is constant.

5. Derive the line equation of the general conic

$$ax^2 + 2hxy + by^2 + 2fy + 2gx + c = 0.$$

6. What is the class of each of the curves in Exs. 3–5?

7. Which of the lines $(2, -1)$, $(3, 2)$, $(-5, 3)$, $(4, 7)$, $(8, 2)$ are tangent to $y^2 = 8x$?

8. Find the tangents to the curve $25x^2 + 16y^2 = 1$ from $(-\frac{1}{25}, -\frac{7}{20})$.

9. Find the equations of the common tangents to the pairs of curves

$$\begin{array}{rrr} 41(x^2 + y^2) = 1 & 2y^2 = x & 25(x^2 + y^2) = 1 \\ 80xy = 1 & 5x^2 + 20y^2 = 4 & 108x^2 + 64y^2 = 3 \end{array}$$

10. A line moves so that the segment intercepted by the axes is constant and equal to a. Find the equation of this line locus. What is the class of the curve? Draw enough of the lines to form an idea of the figure of the curve.

11. Show that the degree of the condition for a point curve to touch a line is the class of the curve. Dualize.

16. Methods of deriving the point equation from the line equation.—The problem of obtaining the point equation of a curve from the line equation is a special case of the problem of envelopes explained in books on the differential calculus.[1] For we are to find the envelope of an arbitrary line of the curve. It is likewise the dual of the problem considered in §15. The method outlined there accordingly may be used to recover the point equation from the line equation. Suppose the line equation is given in the form $\phi(u, v) = 0$. The u-coördinates of the lines common to $\phi = 0$ and an arbitrary point, $ux + vy + 1 = 0$, will be determined by the equation $\phi\left(u, \frac{-1-ux}{y}\right) = 0$. Now the point equation is simply the condition that the point be on the curve or what amounts to the same thing that two of the common lines coincide. This will happen if ϕ considered as an equation in u have a double root. In other words the

[1] See Hulburt, *Calculus*, Chap. 33.

point equation is the u-discriminant of ϕ set equal to zero, and we have to eliminate u between $\phi = 0$ and $\frac{\partial \phi}{\partial u} = 0$.[1]

It is not necessary of course to perform this elimination for every problem. Indeed the discriminant may be calculated once for all for the general equation of a certain degree and thenceforth used as a formula.

17. We shall now explain a **second method** applicable to an important class of curves appropriately defined. In the previous case we began with a relation $\phi(u, v) = 0$ which connects the coördinates of an arbitrary line of the curve. But the curve is equally defined by the *equation* of an arbitrary line of the envelope. For example $x \cos \alpha + y \sin \alpha = p$, α a parameter, is the equation of a family of lines which envelop a circle with center at the origin and radius p.

Consider now the equation

$$t^2x - y + t = 0. \tag{1}$$

If t is given the equation represents a definite line. For varying t it is the equation of a one-parameter family of lines. It therefore defines a line locus. Through an arbitrary point of the plane pass two lines since for given x, y (1) is quadratic in t, *i. e.*, the locus is of the second

[1] If the equation $f(t) = 0$ have a double root r then we may write

$$f(t) = (t - r)^2 . g(t).$$

Now $$\frac{\partial f}{\partial t} = (t - r)^2 \frac{\partial g}{\partial t} + 2g(t) . (t - r) = (t - r) \left\{ 2g + (t - r) \frac{\partial g}{\partial t} \right\}.$$

Hence a double root of $f = 0$ is a simple root of $\frac{\partial f}{\partial t} = 0$. The necessary and sufficient condition that $f = 0$ have a double root is to within a factor the same as the condition that $f = 0$ and $\frac{\partial f}{\partial t} = 0$ have a common root. The respective functions of the coefficients which vanish when and only when these conditions are fulfilled are called the *discriminant* of f and the *eliminant* or *resultant* of f and $\frac{\partial f}{\partial t}$.

class. The envelope of the system is found by expressing the condition that the two lines on a point coincide, or that (1) considered as a quadratic in t have equal roots. The equation of the envelope is therefore

$$4xy + 1 = 0. \qquad (2)$$

Generally, if t is a parameter the equation

$$f(x^{(1)}, y^{(1)}, t^{(n)}) = 0,$$

linear in x and y and of degree n in t represents a singly infinite system of lines which touch a curve of class n. For if t is given $f = 0$ is the equation of a line. But if x and y are fixed $f = 0$ determines n values of t, *i. e.*, on an arbitrary point are n lines of the curve. The point equation is obtained at once by taking the discriminant of f as to t.

In particular if t enters rationally, $f = 0$ is a *rational curve for then the coördinates of the lines of the curve are expressed as rational functions of a single parameter.* This property which is characteristic may be taken as a definition of rational curves.

18. A third form of the problem which can be brought under the second method of solution may be worth mentioning. Suppose the line equation is $\phi(u, v) = 0$ where however u or v can be expressed explicitly as a function of the other. If, *e. g.*, $v = \psi(u)$ the equation of the family of lines becomes

$$f(u) = ux + \psi(u)\, y + 1 = 0$$

and the equation of the envelope is found by eliminating u between $f = 0$ and $\dfrac{\partial f}{\partial u} = 0$.

Example. Let us find the point equation of the curve

$$\frac{1}{u^2} + \frac{1}{v^2} = a^2 \quad \text{(Ex. 10, §15)} \qquad (1)$$

or what is the same thing the envelope of the line $ux + vy + 1 = 0$ where u and v satisfy equation (1).

From (1),

$$v = \frac{u}{(a^2u^2 - 1)^{1/2}} \tag{2}$$

hence the equation of the family of lines is

$$ux + \frac{uy}{(a^2u^2 - 1)^{1/2}} + 1 = 0 \equiv f(x, y, u). \tag{3}$$

$$\frac{\partial f}{\partial u} \equiv x - \frac{y}{(a^2u^2 - 1)^{3/2}} = 0. \tag{4}$$

Eliminating u between (3) and (4) gives the equation of the envelope.

From (4),

$$u = \frac{(x^{2/3} + y^{2/3})^{1/2}}{ax^{1/3}}, \quad (a^2u^2 - 1)^{1/2} = \left(\frac{y}{x}\right)^{1/3}. \tag{5}$$

Substituting in (3)

$$\frac{(x^{2/3} + y^{2/3})^{1/2}}{ax^{1/3}}(x + y^{2/3}x^{1/3}) + 1 = 0. \tag{6}$$

whence

$$x^{2/3} + y^{2/3} = a^{2/3} \tag{7}$$

which is the familiar equation of the *astroid*. Thus the curve (1) of class four is of order six.

19. Fourth method.—In all the preceding cases the solution has depended on the possibility of expressing the line equation as a function of a single parameter of the form $f(x, y, u) = 0$. The point equation was then obtained by eliminating the parameter from $f = 0$ and $\frac{\partial f}{\partial u} = 0$. It is frequently simpler in practice to consider the line equation as a function of two parameters subject to a condition. Thus in the line equation $\phi(u, v) = 0$ the coördinates of the line always satisfy the relation $f \equiv ux + vy + 1 = 0$. Or $f = 0$ will be the equation of an arbitrary line of the curve

if u and v are connected by a relation $\phi = 0$. The point equation may then be expressed by eliminating u and v from $\phi = 0, f = 0$ and $\dfrac{\partial f}{\partial u} \equiv x + y\dfrac{dv}{du} = 0$, where $\dfrac{dv}{du}$ is found by differentiating ϕ.

EXERCISES

1. Plot the system of lines $y = tx + 1/t$. Find the equation of the envelope.

2. Find the locus of a line the sum of whose intercepts on the axis is constant and equal to a. Derive the point equation of the curve.

3. Outline the duals of the methods in §§16–19.

4. Calculate the discriminant of the cubic $ax^3 + bx^2 + cx + d = 0$.

5. Find the line equation of $y^2 = x^3$.

6. Find the line equation of $x^3 + y^3 + 1 = 0$.

7. If a sheet of paper is folded, a corner to an edge, the crease envelopes a parabola.

8. Find the envelope of the normals to the parabola $y^2 = 4ax$. This is the *evolute* of the parabola.

9. Find the point equation of $(a_0t^2 + 2b_0t + c_0)x + (a_1t^2 + 2b_1t + c_1)y + (a_2t^2 + 2b_2t + c_2) = 0$. What are the coördinates of this line?

10. Find the envelope of the line $ux + vy + 1 = 0$ where u and v satisfy the equations $u = \dfrac{3t^2}{t^3 + 1}$, $v = \dfrac{3t}{t^3 + 1}$. Show that this is the same as the point equation of the curve $u^3 + v^3 - 3uv = 0$.

11. Find the line equation of the curve $y = \dfrac{1 - at^2}{t^3}$, $x = ty$.

CHAPTER III

THE LINE AT INFINITY

20. Points at infinity.—The principle of duality breaks down in the elementary geometry of Euclid. Indeed it was necessary to qualify the first theorem of §13. For while two distinct points always determine a line, two lines fail to determine a point when they are parallel. Such exceptions are not uncommon in mathematics but they can frequently be avoided by the aid of appropriate expedients. Often it suffices to modify definitions or merely adopt conventions of language. But sometimes new postulates or assumptions are required. Thus in algebra we might say that the quadratic equation $x^2 - 2ax + a^2 = 0$ has only one root a. For the sake of uniformity however it is customary to say that the equation has *two equal roots*. Here a change of language is all that is needed. On the other hand if the equation $x^2 + x + 1 = 0$ is to have any root it is necessary to extend the domain of numbers to include the imaginary numbers. With these conventions,—that a repeated root counts for two and that imaginary roots are to be accepted equally with real,—we can say *every* quadratic equation has two roots.

Again we might say that a circle cuts a line of its plane in two points, one point or no point. But with the proper modifications we can make the geometry conform to the algebra. Thus a tangent is considered as meeting the curve in "two coincident points." But in order that the statement shall be true universally it is necessary to introduce a new class of points, the "imaginaries." Imaginary points correspond to the imaginary numbers of algebra.

If in solving the equations of line and circle the roots turn out to be imaginary, the points of intersection are said to be imaginary. "No point" is now replaced by "two imaginary points" when without exception *a line cuts a circle in two points*,—real and distinct, coincident or imaginary. The new statement not only serves every purpose of the old but is really more descriptive of the true relation of line and circle.[1]

To say that two parallel lines do not meet is like saying that certain lines have no point of intersection with a circle. There we found that the exception could be removed by introducing imaginary intersection. In an exactly analogous fashion we may introduce a second new class of points into geometry, *points at infinity*, which will serve for the "intersections of parallel lines." We do this formally by means of the following statement which is in the nature of a

Postulate. There is on every line one and only one point at infinity.

21. Some consequences of the new postulate.—It follows at once that *the locus of points at infinity in the plane is a line, the line at infinity*, for it is a figure which every line of the plane cuts in one and only one point. We shall designate the line at infinity by £.[2] This line is not imaginary in the technical sense in which that term is used in mathematics. It is fictitious only in that it differs in some respects from ordinary lines though it enjoys many properties in common with them. Indeed as we shall see, in projective geometry no distinction is made between the line at infinity and other lines.

[1] The axis of a circle is a true example of a line which does not meet the circle. Likewise an element of a circular cone meets the circle of the base in *one* point.

[2] This symbol is made by combining the script £ with the symbol for infinity (∞).

Since there is a line at infinity in every plane *the locus of the infinitely distant points in space is a plane, the plane at infinity.* The line at infinity in any plane is simply the intersection of the plane with the plane at infinity.

Generally *the locus of points at infinity in* S_n *is an* S_{n-1}.

The immediate advantage arising from the recognition of points at infinity is that certain exceptions which occur in duality are removed.[1] We can now say for example that every pair of distinct lines meet in a point. Yet we have done no violence to elementary geometry for with a slight revision of the terminology the familiar properties of Euclid can be described in the new language as well as in the old. We have in fact merely substituted another postulate for the classical parallel postulate. The old definition of two parallel lines as lines (*a*) which lie in the same plane and (*b*) which do not meet however far produced, is replaced by the following: *Two parallel lines are lines which meet at infinity.* This simple statement includes both essential parts of the old definition. A system of parallel lines in the plane or in space meet at a common point at infinity.

Similarly *parallel planes are planes which have their infinitely distant lines in common.*

A line and a plane which are parallel meet at the point at infinity of the line.

That our present machinery is vastly superior to Euclid's in dealing with an important class of theorems on parallels

[1] For other noteworthy exceptions which disappear see below and the following chapter. It is not to be supposed that there are no disadvantages. For example our agreement that there shall be a single infinitely distant point on a line implies that if two points travel in opposite directions on the line they will ultimately arrive at the same destination, *viz.*, at the point at infinity. More properly the line at infinity should be conceived as a repeated line which every line cuts in two coincident points. Even then the points just mentioned and indeed every pair of such coincident points would be infinitely distant from each other. This is however no worse than having distinct points at a zero distance from each other as happens in the case of certain imaginary points.

is evident from the appended exercises. For the proofs here depend on the most elementary notions such as incidence and coincidence, that two points determine a line, three a plane, etc. Whereas the treatment of Euclid is not without its subtleties.

EXERCISES

1. Show that the postulate of §20 in view of our definition of parallel lines is equivalent to the parallel postulate,—through a point there is one and only one line parallel to a given line.

2. Translate into the language of infinite elements the following theorems and prove. The theorems are stated for S_3.

(*a*) Through a point there is a unique plane parallel to a given plane.

(*b*) Two lines which are parallel to a third line are parallel to each other.

(*c*) If a line is parallel to each of two intersecting planes it is parallel to their line of intersection, and conversely.

(*d*) If a line l is parallel to a plane π any plane containing l cuts π in a line parallel to l.

(*e*) Through a given line one plane and only one can be passed parallel to a given skew line.

(*f*) Through a given point one plane and only one can be passed parallel to each of two skew lines.

(*g*) All the lines on a point and parallel to a given plane lie in a plane parallel to the first plane.

(*h*) If a plane contain one of two parallel lines but not the other it is parallel to the other.

(*i*) The intersections of a plane with two parallel planes are parallel lines.

3. Define prism and cylinder in terms of pyramid and cone respectively.

4. Define parallel spaces in S_4. Generalize for S_4 as many of the theorems under Ex. 2 as you can. State some analogous theorems for S_n.

22. The algebraic aspect of the infinite point on a line.— We have seen that there is a certain geometric advantage in the use of elements at infinity. Because of the close connec-

tion between geometry and algebra we should expect a similar algebraic advantage. The intimacy between geometry and algebra is due to the interpretation of numbers as coördinates. Thus along the line we associate with each point a number, and conversely every number determines one and only one point. In order that this correspondence between point and number be without exception it is necessary to attach to the point at infinity on the line a number. To this "improper" point we assign the improper number ∞. We can give any meaning we please to this number provided only the new meaning shall be consistent with the geometric significance already implied.

We saw that parallel lines meet in a point at infinity. Now the line

$$y = tx - a \tag{1}$$

cuts the x-axis at the point

$$x = a/t. \tag{2}$$

The line is parallel to the x-axis when $t = 0$. It will then meet the x-axis in the infinitely distant point. Hence from (2) we have

$$1°.\quad a/0 = \infty, \quad a \neq 0.$$[1]

Similarly the coördinates of the intersections of

$$y = ax^2 + bx + c \tag{3}$$

and the axis of x are the roots of the equation

$$ax^2 + bx + c = 0, \text{ say } x_1 \text{ and } x_2. \tag{4}$$

Now the roots of

$$a + bx + cx^2 = 0 \tag{5}$$

are the reciprocals of the roots of (4), *i. e.*, $1/x_1$ and $1/x_2$. And if $x_1 = 0$, $1/x_1 = \infty$ (from 1°). But the condition that $x_1 = 0$ is $c = 0$. That is, one root of a quadratic equation is ∞ if the coefficient of the highest term is zero.

[1] If $a = 0$ at the same time 1° is indeterminate for then the line coincides with the x-axis and intersects it throughout.

Therefore if $a = 0$, $x_1 = \infty$ (from (4)) and $1/x_1 = 0$ (from (5)). Accordingly we attribute to ∞ the further property

$$2°. \quad 1/\infty = 0.$$

Again if $a = 0$, one root of the equation (4) is $x_1 = -c/b$ which is determinate and not infinite if b, $c \neq 0, \infty$. The other root $x_2 = \infty$. But

$$x_1 + x_2 = -b/a = \infty \tag{6}$$

and

$$x_1 x_2 = c/a = \infty. \tag{7}$$

Hence we have the additional properties from (6)

$$-c/b + \infty = -b/a = \infty \quad (b \neq 0, \text{ i. e., } x_1 \neq \infty)$$

or

$$3°. \quad z + \infty = \infty, \ z \text{ any number not infinite,}$$

from (7)

$$-c/b \cdot \infty = c/a = \infty \quad (c \neq 0)$$

or

$$4°. \quad z \cdot \infty = \infty, \ z \text{ any number not zero.}$$

23. The infinite roots of equations.—Since the equation $ax^2 + bx + c = 0$ has one root equal to infinity when $a = 0$, we may regard the equation $bx + c = 0$ as a quadratic with an infinite root. If in addition $b = 0$ the other root is infinite and the equation $c = 0$ may be looked upon as a quadratic with two infinite roots. These results admit of immediate extension to equations of any degree. For consider the equations

$$f(x) \equiv a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ . \ . \ . \ + a_{n-1} x + a_n = 0$$

and

$$f(1/x) \equiv a_0 + a_1 x + a_2 x^2 + \ . \ . \ . \ a_{n-1} x^{n-1} + a_n x^n = 0.$$

$f(1/x) = 0$ will have one root 0 if $a_0 = 0$, two roots 0 if $a_0 = a_1 = 0$, and r roots 0 if $a_0 = a_1 = a_2 = \ . \ . \ . \ a_{r-1} = 0$.

Under the same conditions $f(x) = 0$ will have one, two, r roots ∞. In other words not only may $ax + b = 0$ be regarded as a quadratic with one root ∞ but as a cubic with two roots ∞ or an n-ic with $n - 1$ roots ∞. Generally, *an equation of apparent degree* r *may be considered as an equation of degree* n *with* n − r *roots* = ∞.

The context will usually indicate when it is desirable to consider the apparent degree of an equation as too low. It will also serve as a guide in selecting the appropriate degree.

The theorem just quoted is invoked in enumerating the intersections of curves when it is desirable to take account of infinite intersections, particularly in the study of asymptotes. For example it is intuitively obvious that a line drawn at random will meet the hyperbola

$$x^2 - 4y^2 = 21 \tag{1}$$

in two points. And experience teaches us that a linear equation and a quadratic like (1) usually have two solutions, though they may be coincident or imaginary. But if we combine the equation

$$x = 2y + 3 \tag{2}$$

with (1) we obtain

$$4y^2 - 4y^2 + 12y = 12. \tag{3}$$

Whence there appears to be a single set of simultaneous values, *viz.*, $x = 5$, $y = 1$, *i. e.*, the line (2) seems to meet the hyperbola in a single point. If however we consider (3) as a quadratic with one root infinite, as is natural since the coefficient of y^2 becomes zero, the exception is avoided and we can say that the line and hyperbola meet in two points, one finite and one infinite. In the same way we should find that the line $x = 2y$ meets the hyperbola in two points at

infinity. With this understanding *every line meets the hyperbola in two points, real or imaginary finite or infinite.*[1]

24. Inconsistent equations in the light of infinite elements.—We shall consider first the linear equations

$$a_1x + b_1y + c_1 = 0 \tag{1}$$
$$a_2x + b_2y + c_2 = 0. \tag{2}$$

The solution of the equations may be written

$$x = \frac{|b_1c_2|}{|a_1b_2|} = \frac{A}{C}, \qquad y = \frac{|c_1a_2|}{|a_1b_2|} = \frac{B}{C}. \tag{3}$$

Obviously there is a unique pair of values of x and y if $C \neq 0$.[2] Let us examine the possibilities when $C = 0$. Two cases present themselves: (I) Either A and B are not both zero or (II) $A = B = 0$. We proceed to the analysis of

CASE I. $C = 0$, $B \neq 0$. Such equations are called inconsistent in elementary algebra and they are said to have "no solution." There are three characteristic hypotheses depending on the way in which C becomes zero.

1°. $a_1 = b_1 = 0$. Then $A = -b_2c_1$, $B = c_1a_2$. Since B is different from zero c_1, $a_2 \neq 0$. Now (*a*) $b_2 = 0$ or (*b*) $b_2 \neq 0$. We shall have respectively

$$\begin{aligned} x &= 0/0 \qquad & x &= \infty \\ y &= \infty \qquad & y &= \infty. \end{aligned} \tag{4}$$

Under our hypothesis the first line on dividing by c_1 reduces to

$$0x + 0y + 1 = 0 \tag{5}$$

whose intercepts $-1/a_1$ and $-1/b_1$ are infinite. The line then may be considered as having moved off to infinity,

[1] These classes are not all mutually exclusive for a finite or an infinite point may be either real or imaginary. The two points need not both belong to the same class and of course they may coincide.

It should be observed that if we adhere to the "one point" "no point" terminology there is no way of distinguishing a line like (2) from a tangent. And "no point" would refer indiscriminately to two imaginary or two infinite points.

[2] A set of values, one for each variable, which satisfy two or more equations is called *a* solution of the equations.

i. e., (5) may be taken as the equation of $\mathfrak{L}$. The second line is parallel (or coincident) with $x = 0$, or not parallel to either axis according as (a) or (b) is fulfilled. In either case the two lines meet at infinity—at the infinitely distant point on the second line.

2°. $b_1 = b_2 = 0$. Then $A = 0$, $B = c_1a_2 - c_2a_1$. The lines reduce to $a_1x + c_1 = 0$, $a_2x + c_2 = 0$. Since $B \neq 0$ both lines exist and remain distinct. They are manifestly parallel to $x = 0$ and intersect therefore at the infinite point of the y-axis whose coördinates are $x = 0/0$, $y = \infty$.

3°. $a_1/a_2 = b_1/b_2 \neq c_1/c_2$. Barring hypotheses 1° and 2° the equations reduce to the form $a_1x + b_1y + c_1 = 0$, $a_1x + b_1y + c_2' = 0$ whose solution is $x = \infty$, $y = \infty$. The lines are parallel and thus meet at infinity.

Under each supposition of Case I we have found that the two equations may be interpreted as representing distinct lines which meet at a single point at infinity. The coördinates of this point will be a solution of the equations and we no longer have pairs of linear equations with no solution. We may retain the term "inconsistent" introducing the following:

Definition.—*The two linear equations are inconsistent when and only when they represent lines which meet at infinity.*

We shall not here delay on Case II. We merely remark that if $A = B = C = 0$ the equations have an infinity of solutions and they are not independent.

25. Three linear equations.—Similarly the solution of the three equations

$$\begin{aligned} a_1x + b_1y + c_1z + d_1 &= 0 \\ a_2x + b_2y + c_2z + d_2 &= 0 \\ a_3x + b_3y + c_3z + d_3 &= 0 \end{aligned} \tag{1}$$

may be written

$$x = \frac{A}{D}, \qquad y = \frac{B}{D}, \qquad z = \frac{C}{D}.$$

If $D = 0$ but not all the numerators are zero the values of one, two or three of the unknowns will be ∞. In that case the equations are "inconsistent" as before. Geometrically, the three planes meet in a unique point on the plane at infinity.

If A, B, C, D are all zero, the equations are dependent and have ∞^1 or ∞^2 solutions according as the planes have a common line or coincide. Such equations may however be inconsistent as for example the equations

$$\begin{aligned} x + y + z &= 1 \\ x + y + z &= 2 \\ x + y + z &= 3. \end{aligned}$$

The three planes are parallel and hence have their infinitely distant lines in common. We are led to extend our definition of inconsistent equations as follows:

Three non-homogeneous linear equations in three variables, whether dependent or independent, are inconsistent when the planes which they represent meet only at infinity.

26. A general definition of inconsistent equations.—There is no difficulty in generalizing these results for sets of n linear equations in n variables. Indeed the definition is valid when there are fewer equations than variables. Thus the equations of two parallel planes would be inconsistent. Moreover any set of equations in linear form will be inconsistent under similar conditions. For example the equations

$$\begin{aligned} ax^2 + by^2 + k_1 &= 0 \qquad\qquad (1) \\ ax^2 + by^2 + k_2 &= 0 \qquad\qquad k_1 \neq k_2 \end{aligned}$$

may be considered linear in x^2 and y^2. Solving we have $x^2 = \infty$, $y^2 = \infty$. The four solutions coincide in pairs, *i. e.*, the curves have double contact at infinity.

Two equations need not be of the same degree to be

inconsistent. We had an instance of this in §23. As another example we mention the equations

$$\begin{aligned} x^2 - y^2 + y - 1 &= 0 \\ x^2 + x^2y - y^3 &= 0. \end{aligned} \tag{2}$$

Substituting in the second equation the value of x^2 obtained from the first we find

$$0y^3 + 0y^2 + 0y + 1 = 0,$$

all of whose roots are infinite. The two curves intersect wholly at infinity. We can now formulate our general

Definition.—*r non-homogeneous equations in n variables, $r \leq n$, are inconsistent when and only when the graphs represented have all their intersections whether real or imaginary at infinity.*[1]

Enough has been said to demonstrate the utility of infinite elements in algebra as well as geometry. Another service which they perform conducive to generality of statement has been anticipated in §23. There it was possible to say that every line meets an hyperbola in two points,—surely a verbal economy. Likewise theorem 4′ (§13) that two curves of orders m and n have precisely mn points in common is valid only when infinite as well as coincident and imaginary points are taken into account.

EXERCISES

1. What is the dual of the origin? State and prove the dual of the theorem: If the equation of a point locus contain no constant term, the origin is a point of the locus (curve).

2. Find all the intersections of $x^3 - y^3 = 19$, $x - y = 1$.

3. Determine a and b so that the equation

$$x^2 - (ax + b)x - 2x - 2(ax + b) + 1 = 0$$

shall have ∞ for both its roots.

[1] An algebraic criterion would be better if we were discussing the equations solely but we are interested here in the geometrical aspect of the case. Furthermore the very purpose of this exposition is to show that with the admission of infinite elements inconsistent equations so called disappear as an exceptional class and we have little occasion to recognize them. However failure to obtain finite solutions is usually a satisfactory algebraic test of inconsistence.

4. Show by the distance formula that all lines of the system $x \pm iy + k = 0$ ($i^2 = -1$, $k \neq 0$) are at an infinite distance from the origin. Reconcile this with the statement that there is just one line at infinity.

5. Dualize Case I (§24), indicating the geometrical relation of the two points under each hypothesis.

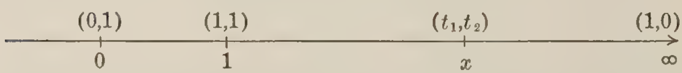
6. Analyze Case II and show that (*a*) the lines coincide either as finite lines or with the line at infinity, (*b*) one line vanishes identically, (*c*) both lines vanish identically.

7. If $D = 0$, A, B, C not all zero (§25), what is the relation of the three planes? Three answers.

8. Make a list of the exceptions that have been removed already by the recognition of infinite elements.

27. Homogeneous coördinates on the line.—When the infinite elements come into consideration it is usually preferable to employ *homogeneous coördinates.* In this system we associate with each point along a line *two* numbers or coördinates t_1, t_2. The new coördinates are connected with the old by means of the relation (1) $x = t_1/t_2$. From this equation it is plain that not both of the homogeneous coördinates of a point can be zero else the point is indeterminate. Again if the t's are given x is uniquely determined. On the other hand if x is given only the *ratio* of the t's is known. In other words (rt_1, rt_2) defines the same point as (t_1, t_2).

If $t_1 = 0$, $x = 0$; if $t_2 = 0$, $x = \infty$ and if $t_1 = t_2$, $x = 1$. That is, the coördinates of the origin, the point at infinity and the unit point respectively are $(0, t_2)$, $(t_1, 0)$, (t_1, t_1) or $(0, 1)$, $(1, 0)$ and $(1, 1)$ since only the ratios are required.



In the non-homogeneous system the origin and unit point may be chosen arbitrarily, *i. e.*, the numbers 0 and 1 may be attached to any two points whatever. The number ∞ however is always, at least tacitly, assigned to the point at

infinity. The choice of the three points which are to be designated by 0, 1 and ∞ completely establishes the coördinate system. The homogeneous system here introduced is subject to similar restrictions, for we are dealing with *Cartesian* coördinates in both systems.

An equation $f(x) = 0$ by means of the relation (1) is transformed into an equation homogeneous[1] in t_1, t_2 *and of the same degree.* While an equation in the t's goes back into an equation in x by means of the substitutions $t_1 = x$, $t_2 = 1$. As an example the cubic $ax^3 + bx^2 + cx + d = 0$ becomes

$$at_1^3 + bt_1^2t_2 + ct_1t_2^2 + dt_2^3 = 0.$$

If $a = 0$,	$t_2 = 0$,	$x = \infty$,
$a = b = 0$,	$t_2^2 = 0$,	$x = \infty, \infty$,
$a = b = d = 0$,	$t_1t_2^2 = 0$,	$x = 0, \infty, \infty$,
$a = d = 0$,	$t_1t_2 = 0$,	$x = 0, \infty$.

This illustrates the superiority of the homogeneous form in discussing the infinite roots of equations. Not only do the infinite roots appear more readily, being placed on the same basis as zero roots, but the degree of the equation is always in evidence.

28. Homogeneous coördinates in the plane.

In the plane the homogeneous coördinates of a point are three numbers, not all zero (x', y', z') connected with the Cartesian coördinates by the equations

$$x = x'/z',\ y = y'/z'.$$

Dually the homogeneous cöordinates of a line are three numbers not all of which are zero, connected with the Plücker coördinates by the equations

$$u = u'/w',\ v = v'/w'.$$

[1] Hence the name "homogeneous coördinates." There are many systems of homogeneous coördinates on the line, depending on the geometrical interpretation of the numbers. In the present system the number x or the ratio of the t's represents a directed segment. The absolute value of the number gives the length of the segment and the sign indicates the direction from the origin.

As before only the ratios of x', y' and z' are determined when the point is given by its Cartesian coördinates. Thus $(kx', ky', kz') \equiv (x', y', z')$.

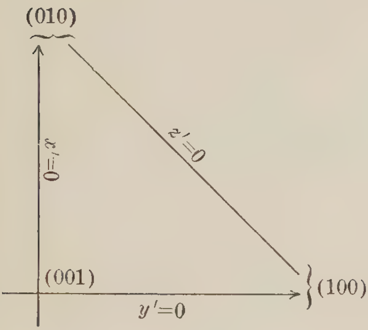
If (u', v', w') are given the line is uniquely determined. But if the line is given only the ratios of u', v' and w' are known for $(ku', kv', kw') \equiv (u', v', w')$.

An equation $f_n(x, y) = 0$ under the substitutions above is transformed into an equation homogeneous in x', y', z' of the same degree. The resulting homogeneous equation can be written down most readily by simply inserting in each term the requisite power of z to bring it up to the nth degree. The Cartesian equation is recovered by substituting $x' = x$, $y' = y$, $z' = 1$.

If $y' = 0$, $(z' \neq 0)$, $y = 0$ and the point lies on the x-axis. If $z' = 0$ at the same time $x = \infty$, y is indeterminate and the point is still on the x-axis though at infinity. Hence $y' = 0$ may be called the *equation of the x-axis in homogeneous coördinates.* Likewise $x' = 0$ is the equation of the y-axis. When $z' = 0$ either $x = \infty$ or $y = \infty$ or both and the point is on $\mathcal{L}$. That is $z' =$

If $u' = 0$, $(w' \neq 0)$, the x-intercept $x_0 = \infty$ and the line is parallel to the x-axis, *i. e.*, is on the infinitely distant point of the x-axis. If $u' = 0$ and $w' = 0$ then $v = \infty$ and u is indeterminate, *i. e.*, the y-intercept $y_0 = 0$ and x_0 is arbitrary hence the line is the x-axis. When $u' = 0$ therefore the line always contains the point at infinity on the x-axis. Hence $u' = 0$ *is the equation*

0 is the equation of the line at infinity.



These three lines

$$x' = 0, y' = 0, z' = 0$$

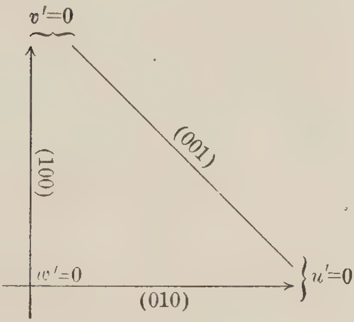
may be called a triangle[1] of reference and are designated x-, y- and z-axes respectively. The cöordinates of its vertices are

$$(x', 0, 0), (0, y', 0), (0, 0, z')$$

or since only the ratios are

of the infinite point of the x-axis. Similarly $v' = 0$ is the equation of the infinitely distant point of the y-axis.

When $w' = 0$ either $u = \infty$ or $v = \infty$ or both, *i. e.*, $x_0 = 0$ or $y_0 = 0$ or both and the line is on the origin. Hence $w' = 0$ is the equation of the origin.



These three points

$$u' = 0, v' = 0, w' = 0$$

may be called a triangle of reference. The coördinates of its sides are (1, 0, 0), (0, 1, 0), (0, 0, 1). The first two are the Cartesian axes, the third is the line at infinity.

[1] The coördinates on that account are sometimes termed *trilinear coördinates.* In practice primes are dropped since there is little danger of confusion. Observe that now the x-axis is the line whose equation is $x = 0$, a custom that unfortunately is not followed in elementary analytics.

essential (1, 0, 0), (0, 1, 0), (0, 0, 1). The first two are the infinite points of the Cartesian axes, the third is the Cartesian origin.

The incidence condition of point and line becomes in homogeneous coördinates

$$(ux) \equiv ux + vy + wz = 0.^1$$

This equation therefore represents the line (u, v, w) or the point (x, y, z) according as the point or line coördinates are considered variable. Thus the coördinates of a point or line are simply the coefficients in its homogeneous equation,—or three numbers proportional to the coefficients.

The equation of the line on the two points (x_1, y_1, z_1), (x_2, y_2, z_2) is

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0.$$

The equation of the point of intersection of the lines (u_1, v_1, w_1), (u_2, v_2, w_2) is

$$\begin{vmatrix} u & v & w \\ u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \end{vmatrix} = 0.$$

The equation of the line on the points

$$a_1u + b_1v + c_1w = 0$$
$$a_2u + b_2v + c_2w = 0$$

is

$$\begin{vmatrix} x & y & z \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0,$$

or $|bc|x + |ca|y + |ab|z = 0$.

The equation of the common point of the lines

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$

is

$$\begin{vmatrix} u & v & w \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0,$$

or $|bc|u + |ca|v + |ab|w = 0$.

[1] (ux) is a convenient symbol for a homogeneous function linear in two sets of variables. Likewise (ax) might denote either $ax + by + cz$ or $a_1x + a_2y + a_3z$, the precise coefficients being immaterial.

That is, the coördinates of the line on two points are proportional to the determinants of the matrix

$$\begin{Vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{Vmatrix}$$

formed by the coefficients in the equations of the points.

Hence the coördinates of the point of intersection of two lines are proportional to the determinants of the matrix

$$\begin{Vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{Vmatrix}$$

formed by the coefficients in the equations of the lines.

In solving a system of n equations homogeneous in $n + 1$ variables only the ratios of the variables can be determined. Two methods are available. The equations may be divided by one of the variables say t and treated just as non-homogeneous equations in x/t, y/t, z/t, etc. The numerators of these fractions are then taken as the values of x, y, z, etc., and the denominator is the value of t. Or n of the variables may be expressed in terms of the remaining one to which an arbitrary value is assigned. If in either case the isolated variable turns out to be zero it is usually necessary to select another which is always possible since at least one of the variables must be different from zero. A set of equations in the linear form however can always be solved by the determinant method even when some of the values are zero. The system of values then consists of $n + 1$ numbers proportional to properly chosen determinants of the matrix made up of the coefficients in the equations.

EXERCISES

1. Find the condition that the three points (a) (x_i, y_i, z_i), (b) $a_iu + b_iv + c_iw = 0$, $i = 1, 2, 3$, be on a line. Dualize.

2. Find the coördinates and the equations of the lines on the pairs of points $(2, 3, 4)$, $(\frac{1}{2}, \frac{1}{3}, \frac{1}{4})$; $(1, 2, -7)$, $(5, -2, -11)$; $(1, 1, 2)$, $(1, 1, 4)$; $2u - v + 5w = 0$, $3u + 2v - w = 0$; $4u - 3v - 8 = 0$, $3u - 5v - 6 = 0$.

3. Write the equations of the six lines joining the four points $(1, \pm 1, \pm 1)$.

4. Find the coördinates and equations of the points of intersection of the pairs of lines $x - y = 6z$, $3x + 4y = 4z$; $2x - y + z = 0$, $4x - 3y - 2z = 0$; $(2, 3, -7)$, $(3, -2, -4)$; $(1, 2, 4)$, $(2, -1, 3)$.

5. Show that the lines $3x - 2y - 2z = 0$, $3x - y + 11z = 0$, $x - y - 5z = 0$ meet in a point. Find the coördinates and the equation of the point.

6. Find the value of a such that the points $3u - 2v = 0$, $u - v + 2w = 0$, $3u - v + aw = 0$, lie on a line.

7. Write the equations of the lines joining the vertices of the triangle of reference to the point (a, b, c). Find the coördinates of the points in which these lines cut the sides of the triangle. Write the equations of the sides of the new triangle. Show that the sides of the two triangles meet in pairs in three points of a line. Write the equation of this line. This is the *polar line* of (a, b, c) with respect to the triangle.

8. What is the polar line of the centroid (intersection of medians) of a triangle?

9. Solve the system of equations $2x + 3y - 2z + 4t = 0$, $x + y + z - 5t = 0$, $3x + 5y - 5z - 10t = 0$.

10. Solve the system $x + y - z + 2t = 0$, $2x - 3y - 5z + 4t = 0$, $3x - 2y + z + 6t = 0$.

29. Classification of conics.—An equation $f_n(x, y) = 0$, homogeneous and of the nth degree in x and y, breaks up into n factors of the form $ax + by$. The equation thus represents n lines on the point $(0, 0, 1)$. If therefore in the homogeneous equation of a curve $f_n(x, y, z) = 0$ we set $z = 0$ we obtain the equation of the n lines from the vertex w to the points of intersection of the curve and the z-axis ($\mathcal{L}$). The coördinates of the points of intersection are the simultaneous solutions of the system $f = 0$, $z = 0$. Similar statements of course hold for the other axes.

Dually an equation $f_n(u, v) = 0$, homogeneous in u and v represents n points on the junction of $u = 0$, $v = 0$, *i. e.*, on $(0, 0, 1)$. Thus if we put u, v or $w = 0$ in the homogeneous equation of a curve of class n, $f_n(u, v, w) = 0$, we

obtain the equation of the n points on the x-, y-, or z-axis respectively cut out by the tangents to f from the u-, v-, or w-vertex. The coördinates of the n tangents are the simultaneous solutions of $f = 0$ with either u, v or $w = 0$.

We are now in a position to study most conveniently the behavior of a curve in the infinite region. For example the homogeneous equation of the parabola $y^2 = 4ax$ is $y^2 = 4azx$. Now the parabola is tangent to the line $x = 0$ at the point where this line meets $y = 0$. But since the equation is symmetrical in x and z the parabola will have the same relation to $z = 0$ as to $x = 0$. *Hence the parabola is tangent to the line at infinity at the infinite point of its axis*, $y = 0$.

The general equation of the second degree in three homogeneous variables may be written,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \tag{1}$$

Such an equation defines a *conic*. The coördinates of the points in which this curve cuts the line at infinity are found by solving (1) with $z = 0$. We obtain thence

$$ax^2 + 2hxy + by^2 = 0, \text{ or } a(x/y)^2 + 2hx/y + b = 0 \tag{2}$$

as an equation to determine the ratios of x to y. However we are interested not in the actual points of intersection but in the relation of conic and line. Now the nature of the roots of a quadratic depends upon the value of the discriminant D. The roots of (2), *i. e.*, the points in which the conic cuts the line will be real and distinct, real and equal or conjugate imaginary according as [1]

$$D \equiv ab - h^2 < 0, = 0, > 0. \tag{3}$$

But these are just the conditions that a proper conic be a hyperbola, a parabola or an ellipse.

[1] Here as elsewhere, unless the contrary is specifically stated. the coefficients are assumed to be real.

The behavior of conics with respect to $\mathcal{L}$ therefore serves as a basis of classification into the three principal types for we may say,

A proper conic is a hyperbola, a parabola or an ellipse according as it meets the line at infinity in points which are real and distinct, real and concident or conjugate imaginary.

30. Degenerate conics. The line at infinity not a part of the locus.—When a conic breaks up into two lines, not including $\mathcal{L}$, it can be shown without difficulty that the lines are

1°. $D < 0$, real and distinct,
2°. $D > 0$, conjugate imaginary,[1]
3°. $D = 0$, (*a*) real parallel lines; (*b*) real coincident lines.

Since these line pairs meet the line at infinity exactly as do the proper conics we shall call them degenerate hyperbolas, ellipses and parabolas respectively.

For the case where the line at infinity is a part of the locus see below (§35).

31. The aspect of any curve is affected to an extraordinary degree by the way it gets off to infinity. Indeed the behavior of curves at infinity has been used extensively as a basis for their classification. As an illustration of the first remark consider the equation of the conic $y'^2 = 4az'x'$. This equation may be made non-homogeneous by setting $z' = 1$. In effect this is simply selecting the line $z' = 0$ to be $\mathcal{L}$ and the curve is the familiar parabola.

If however we wish to send $y' = 0$ off to infinity, regarding $x' = 0$, $z' = 0$ as the Cartesian axes it is only necessary

[1] A line is *imaginary* when, the triangle of reference being real, its equation necessarily contains one or more imaginary coefficients. Two imaginary lines are *conjugate* if their equations are interchanged when i is replaced by $-i$, $i = \sqrt{-1}$.

Ex. Two conjugate imaginary lines meet in a real point.

to substitute $y' = 1$, $x' = x$, $z' = y$. The equation then becomes $4axy = 1$, a rectangular hyperbola.

In a sense there is no difference in the two curves since their homogeneous equations are virtually identical. The great difference in the figures depends on the isolation of different lines to be $\mathcal{L}$.

32. Definition of asymptote.—Consider next the relation of an hyperbola to its asymptotes. Let the equation of the hyperbola in homogeneous coördinates be

$$b^2x^2 - a^2y^2 = cz^2. \tag{1}$$

The asymptotes of this hyperbola are

$$b^2x^2 - a^2y^2 = 0, \text{ or } bx \pm ay = 0. \tag{2}$$

Solving the equation of either of the lines (2) with (1) we have $z^2 = 0$. Hence each line (2) meets the curve in coincident points. That is, *the hyperbola is tangent to its asymptotes at infinity.*

The equation

$$(ax)(bx) = kz^2, \tag{3}$$

where the coefficients on the left are fixed constants and k a parameter, defines a one-parameter family of hyperbolas since obviously every curve meets $\mathcal{L}$ in two real, distinct points. Moreover each curve of the system is tangent to each of the lines $(ax) = 0$, $(bx) = 0$ at infinity. In other words (3) is a family of hyperbolas with these lines for common asymptotes. But since all members of the family touch the same lines at the same points they are tangent to each other at those points. Hence (3) *represents a system of conics which have two real contacts at infinity with the line at infinity as the common chord.*

Equation (3), containing five essential constants, is a form to which the equation of any hyperbola can be reduced.

It follows that the equation of any hyperbola in Cartesian coördinates, rectangular or oblique, can be written

product of asymptotes = a constant.

By analogy *we define an asymptote of any curve as a tangent line whose point of contact is on the line at infinity.* An asymptote need not be a simple tangent as in the case of the hyperbola but may meet its curve in any number of coincident points up to the degree of the curve. Neither do we restrict asymptotes to real lines preferring to speak of real or imaginary asymptotes.

33. The circular points.—But the relation of a circle to the line at infinity is from the present point of view of paramount significance and importance. The general equation of a circle in homogeneous coördinates may be written

$$k(x^2 + y^2) + 2fyz + 2gzx + cz^2 = 0. \qquad (1)$$

We shall say that this equation defines a circle when the coefficients are any complex numbers. If $k \neq 0$ the circle will cut the line at infinity in points given by

$$x^2 + y^2 = 0, \text{ or } (x + iy)(x - iy) = 0. \qquad (2)$$

We have thus as the coördinates of intersection $(1, i, 0)$, $(1, -i, 0)$. If $k = 0$ the line at infinity is a part of the locus which contains the points as before. But since (1) is every circle in the plane we have the remarkable theorem

All circles in the plane pass through the same two conjugate imaginary points at infinity.

Conversely any conic, real or imaginary, which contains these points is a circle. For if the conic ((1) §29) is on the points we must have

$$\begin{aligned} a + 2ih - b &= 0 \\ a - 2ih - b &= 0 \end{aligned} \qquad (3)$$

whence, adding and subtracting, $h = 0$, $a = b$. Q.E.D.

In particular if a, b, h are real a conic which contains one of the points contains both, for then either equation (3) requires $h = 0$, $a = b$.

Thus the necessary and sufficient condition for a conic to be a circle is that it be on these points.

Because of their connection with circles these two points, frequently designated by I and J, are called the *circular points*. Taken together they constitute the *absolute* so named by Cayley. The absolute is thus a degenerate line conic. Its point equation, which must be of the second order, is obviously $z^2 = 0$. In other words the absolute as a locus of points is a real line, the line at infinity, repeated.[1]

We now see that many exceptional properties of circles are only apparent. Thus while two conics ordinarily meet in four points two circles appear to meet in only two. This is because two intersections are already preempted for the circular points. Likewise a circle can be made to pass through three arbitrary points only since it must first pass through I and J. Indeed many properties of circles can be translated into properties of conics on two fixed points.

34. Isotropic lines.—The two conjugate imaginary lines which connect a point with the circular points are termed the *circular rays* from the point. Thus $x + iy = 0$, $x - iy = 0$ are the circular rays from the origin. More generally any line on either of the circular points is called an *isotropic* line. That is, isotropic lines are the lines (tangents) of the absolute considered as an envelope.

Isotropic lines reveal curious and startling properties when tested by the familiar analytic formulas for real lines. We mention the following which can be verified by means of the equation

$$y = \pm ix + k$$

which represents all isotropic lines.

[1] *Cf.* p. 33, footnote.

The distance between any two points on an isotropic line is zero,[1] *unless one of the points is a circular point when the distance is arbitrary.*

The distance from an arbitrary point to an isotropic line is infinite unless the point is on the line when the distance is arbitrary.

Every isotropic line is perpendicular to itself.

An isotropic line makes an infinite angle with every real line. For let t be the slope of the real line and θ the angle it makes with an isotropic line. Then

$$\tan\theta = \frac{(t \pm i)}{(1 \mp it)} = \pm i.$$

Expanding by McLaurin's theorem

$$\theta = \tan^{-1} \pm i = \pm i(1 + \tfrac{1}{3} + \tfrac{1}{5} + \ . \ . \ .) = \infty.$$

These properties may be rejected of course on the ground that the formulas apply only to real lines. Or they may be regarded as implying new attributes of the concepts distance, perpendicular, angle.

35. Classification of circles.—Since a circle is a conic on I and J it is geometrically evident that we have the following types. If the line at infinity is not a part of the locus we have (1°) a proper circle, (2°) a pair of circular rays or *null circle.* If the line at infinity is a part of the locus we have three additional varieties: (3°) $\mathcal{L}$ and an arbitrary line (not isotropic), (4°) $\mathcal{L}$ and any isotropic line. (5°) $\mathcal{L}$ repeated.

If $\triangle$ denote the discriminant of the circle § 33 and r the length of the radius we have

$$\Delta = k(kc - f^2 - g^2),\ \ r^2 = \frac{f^2 + g^2 - kc}{k^2},$$

coördinates of center, $(-g/k,\ -f/k)$. The five classes may then be characterized as follows:

[1] For that reason isotropic lines are sometimes called *minimal* lines.

1°. $\Delta \neq 0$, proper circle, center and radius finite.

2°. $k \neq 0$, $kc - f^2 - g^2 = 0$, null circle, center finite, radius zero.

3°. $kc - f^2 - g^2 \neq 0$, $k = 0$, center at infinity, radius infinite.

4°. $k = kc - f^2 - g^2 = 0$, f and g not both zero, center at infinity, radius indeterminate.

5°. $k = f = g = 0$, $c \neq 0$, center and radius arbitrary.

The degenerate circles appear as limiting cases in families of proper circles. For example in the equation $x^2 + y^2 = a$ if a changes from a positive number through zero to a negative number the circle changes from a real proper curve through a null circle to a proper imaginary circle.[1]
If the other coefficients remain fixed in (1) §33 and k decreases to zero the center recedes to infinity and the circle approaches type 3°. On the other hand if k tends to infinity the center moves to the origin and the circle becomes a null circle.

EXERCISES

1. Find the homogeneous coördinates of the points of intersection of the central conic $ax^2 + by^2 + c = 0$ with $\mathfrak{L}$.

2. Find the homogeneous coördinates of the points of intersection of $xy = a$ with the rectangular axes, *i. e.*, with its asymptotes.

3. Find the intersections of the family of conjugate hyperbolas $x^2/a^2 - y^2/b^2 = \pm k$, k the parameter, with their asymptotes.

4. How many asymptotes may a curve of degree n have? What are the asymptotes of a circle, an ellipse? What becomes of the asymptotes of the parabola?

5. Show that the family of circles $x^2 + y^2 = a$ are tangent to the circular rays $x \pm iy = 0$ at I and J, thus proving that concentric circles have double contact at the circular points.

6. Find the intersections of the cissoid $y^2(2a - x) = x^3$ with the line at infinity.

[1] Imaginary since all its points are imaginary even though the coefficients in the equation are real. Proper because it does not decompose into lines.

7. What are the intersections of the folium of Descartes $x^3 + y^3 - 3axy = 0$ with the line at infinity? How many real asymptotes?

8. Study the behavior of the cubical and semi-cubical parabolas $y = x^3$, $y^2 = x^3$ at the origin and at infinity. Hence compare the two curves. Also compare the curves with $x^2y = 1$. Draw figures of each.

9. Study the behavior of the witch $y(x^2 + 4a^2) = 8a^3$ at infinity. Suggestion: Send $x = 0$ and $y = 0$ successively off to infinity and compare the curve with those in Ex. 8.

10. Given a circle with center at the origin, radius r and a fixed tangent parallel to the axis of y. Let a variable tangent with contact at a point P cut the fixed tangent at Q. Show that the locus of the mid-point of the segment PQ is a cubic curve on I and J.

11. Dualize the example of §29 or otherwise discuss the behavior of $u^2 = 4av$ at the origin and at infinity.

12. If $C = 0$ is the equation of any proper circle, $(ax) = 0$ any line and $L = 0$ the equation of the line at infinity show that the equation of every circle can be written $C + kL(ax) = 0$.

13. Write the equation of the pair of points I, J, *i. e.*, of the absolute.

14. Find the common lines of the absolute and the parabola $y^2 = 4x$ and show that they meet at the focus.

15. Find the common lines of the absolute and (a) $xy = k$; (b) $x^2/a^2 \pm y^2/b^2 = 1$.

16. Show that the tangents from I and J to the ellipse, Ex. 15, meet in the four points $(\pm c, 0)$ and $(0, \pm ic)$ where $c^2 = a^2 - b^2$. These latter points are the *imaginary foci* of the ellipse. Find the four foci of the hyperbola in Ex. 15.

17. Prove that the sum of the focal radii drawn from any point on an ellipse to the imaginary foci is constant and equal to $2b$. Prove a similar property for the imaginary foci of the hyperbola.

18. How many conditions on a curve to be given (a) an asymptote; (b) the direction only of an asymptote? Why?

19. If both the asymptotes of a conic are given what sort of family is determined?

20. Find the equation of the hyperbola with the asymptotes $x + y - 1 = 0$ and $2x - y + 1 = 0$ and passing through the point $(2, 3)$.

21. Recalling our definition of a parabola, how many parabolas can be drawn tangent to four lines, *i. e.*, inscribed in a quadrilateral? (Consider the conics as given in lines.)

22. How many circles can be drawn to touch three lines?

23. Show that $y = \frac{ax + b}{cx + d}$ is a rectangular hyperbola.

24. Show that the angle between two isotropic lines, one on I and one on J, is infinite.

25. Find the equation of the null circle on each of the points $(0, 0)$, $(-3, -2)$, $(-5, 4)$.

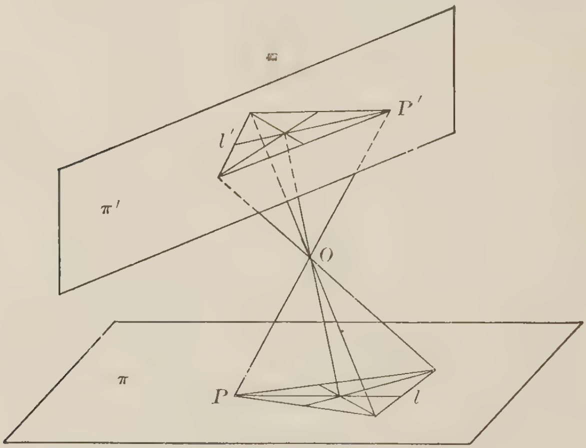
26. Show that every curve of odd order with real coefficients has at least one real point at infinity, *i. e.*, has at least one infinite branch.

27. Trace the changes in the system of hyperbolas $9x^2 - 16y^2 = a$ as a varies from a positive to a negative number through zero.

CHAPTER IV

PROJECTIVE PROPERTIES. DOUBLE RATIO

36. Projection.—The *elements* most commonly employed in geometry are the linear spaces, points, lines, planes, etc. Any collection of such spaces is a *geometrical figure* or briefly a figure. Confining ourselves to ordinary space let us consider a figure F made up of points and lines lying in



a plane π. If O is a point not in π then every point of F determines with O a line and every line of F determines with O a plane. We have thus a space figure S consisting of lines and planes on O. This process by which the conical figure S is obtained from F is called *perspection.* Any *section* of this space figure by a second plane π' not on O is a new plane figure F'. The process by which F' is derived from F is

termed *projection,* or more specifically *central projection, O* being the *center.* Thus a projection is a perspection followed by a section. Obviously if we began with F' we could recover F by a process of projection. Hence *either figure is a projection of the other from O.*

Projective geometry is the study of those properties which are common to a figure and its projection. Such properties are called *projective.*[1] Or we may say that a projective property is one that is not affected by projection.

37. One-to-one correspondence.—Let us consider the relation between F and F' above. For every point P in F there is a corresponding point P' in F', *viz.*, the point in which OP pierces π', and reciprocally. Likewise to every line l in F corresponds a line l' in F', namely the intersection of the plane Ol with π', and conversely. This is expressed by saying that between a figure and its projection there exists the relation of *one-to-one correspondence.*

The idea of one-to-one correspondence (notation (1, 1) correspondence) is so important that we define it formally. *Two classes of elements are in* (1, 1) *correspondence if every element in either class corresponds to one and only one element of the other.* The elements in the two classes need not be of the same kind. Thus we may have a (1, 1) correspondence between points of one figure and lines or planes of another. Among familiar examples of (1, 1) correspondence may be mentioned that between

(*a*) the points of a line and numbers of a coördinate system

(*b*) points of a curve and lines of the dual curve

(*c*) points of the plane and lines of the plane[2]

(*d*) circles in a plane and points of ordinary space

(*e*) conics in a plane and points of S_5

[1] By the older writers *descriptive.* According to present usage descriptive geometry is quite another science.

[2] In virtue of the equation $ux + vy + wz = 0$.

38. Metric and projective properties.—We shall notice first some non-projective properties. Obviously the distance between two points and the angle between two lines are altered in projection, *i. e., neither the size nor the shape of a figure is preserved.* Since angles are changed parallel lines in general go into intersecting lines so that projective geometry has no concern with the troublesome question of parallelism which distressed the followers of Euclid for centuries. The line at infinity projects into an ordinary line. It therefore loses all its peculiarities and is treated exactly as other lines. Since a circular cone is merely a perspection of a circle from the vertex it follows that any section, not on the vertex *i. e., any proper conic is a projection of the circle.* Accordingly we recognize but one species of proper conic in projective geometry.

It is evident that there must be a vast difference between the geometry we are considering and the familiar geometry of Euclid. For it is fundamental in Euclid that a figure can be freely moved without changing either its size or its shape. Without this axiom it might seem that the whole temple of geometry would be demolished. On the contrary we shall see that not only are certain properties invariant under projection but that they are after all the salient properties,—that indeed Euclidean geometry is only a special case of projective.

What then are some of these projective properties? First of all the nature of the element is not changed, *i. e.*, a point projects into a point and a line into a line. Moreover the points of a line in F go into points of the corresponding line in F'. Thus if three, or any number of points, in F lie on a line the projections of those points will lie on a line in F'. Dually if a set of lines of F meet in a point the projections of those lines will meet in a point of F'. The vertices and sides of a triangle go into the vertices and sides of a

triangle though of course the area of the triangle is altered. A conic is projected into a conic, *i. e.*, the order of the curve is not disturbed. If a line l is tangent to a curve C at a point P then l' is tangent to C' at the point P'.

Thus projective geometry deals with those relationships which do not involve magnitudes. Those geometries like Euclid which are concerned with the comparison of magnitudes,—lengths, areas, volumes etc.,—and therefore with measurement are called *metric*. They may be described as quantitative while projective geometry is essentially qualitative.

The instruments of construction in Euclid are a ruler, for drawing straight lines, and a compass for describing circles, or what amounts to the same thing for measuring distances. In projective geometry on the other hand, since distances are changed, we use only the ruler, *i. e.*, a straight-edge. To solve a problem by means of one geometry or the other is to solve it with (only) the instruments characteristic of each. Thus while it is impossible to trisect an angle or duplicate a cube by Euclid both problems admit of simple solution by the use of conics and curves of higher order.[1]

39. Metric and projective properties in their relation to the absolute.—The interrelation between projective and metric geometry will be pointed out repeatedly. But that the distinction may always be clear we give another criterion for differentiating between projective and metric properties. In (Euclidean) metric geometry the line at infinity is

[1] The student should convince himself that practically the whole of elementary geometry is metric. Thus perpendicularity involves the notion of *equal* angles, circle that of length; parallelism is tied up with the angle sum of a triangle, etc. Between the geometry of Euclid which preserves both the size and shape of figures and projective geometry which preserves neither is a geometry which preserves shape, namely the geometry of similar figures or *equiform* geometry. Its characteristic instruments are the straight-edge and the pantograph.

isolated and invested with peculiar properties, as, *e. g.*, those concerning distance and angle. In projective geometry it is treated like any other line. Such metrical ideas as parallel lines, circle, parabola, asymptote, focus were defined in the language of infinite elements. But when the line at infinity is projected into an ordinary line a parabola for example loses its distinctive features and becomes merely a conic tangent to a line. Hence *if a geometrical statement involve the line at infinity, i. e., have a special relation to the absolute, it is metric, otherwise projective.*

The absolute thus not only furnishes a convenient basis of distinction between projective and metric properties but the transition from one to the other can actually be effected through its mediation. A projective theorem concerning any plane figure can be translated into a Euclidean theorem by isolating a line for $\mathfrak{L}$ or a pair of points for I and J. Indeed several metric theorems can be obtained by selecting different lines for $\mathfrak{L}$. And certain Euclidean theorems can be stated projectively by considering I and J as an ordinary point pair or $\mathfrak{L}$ as an ordinary line.

EXERCISES

Exercises 1–3 refer to §36

1. Construct in each of the planes π and π' the line corresponding to the line at infinity in the other.

2. Construct in π' the point corresponding to the intersection of two parallel lines in π.

3. Show that the (1, 1) correspondence between π and π' would break down were it not for infinite elements. Show also that, assuming the parallel postulate, the (1, 1) correspondence requires that there be a *line* at infinity.

4. Recalling that in space a point and plane are dual and a line is self-dual state the duals of the definitions of (*a*) a plane figure, (*b*) perspection, (*c*) section.

5. If two triangles (in different planes) are so situated that lines joining corresponding vertices meet at a point, the intersections of corresponding sides lie on a line. State the space dual.

6. Translate into projective theorems where the line at infinity is an ordinary line

(*a*) The locus of the foci of parabolas on three points is a quintic curve passing through I and J.

(*b*) The circle circumscribing the triangle formed by any three tangents to a parabola passes through the focus.

(*c*) The common chords of three circles taken in pairs meet in a point.

(*d*) The foci of the five parabolas each of which touches one set of four out of five lines lie on a circle.

40. Perspective and projective figures in one dimension.—We shall take up now the special case of projection in a plane together with the dual, exhibiting dual theories in parallel columns.

The set of all points on a line is called a *pencil* or a *range* of points or briefly a range. The line itself is called the *axis* of the range. In particular any set of collinear points finite in number is called a range.

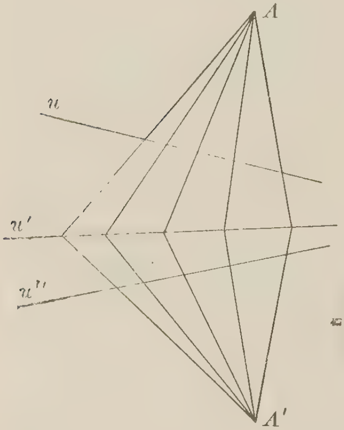
The set of all lines on a point is called a *pencil* of lines or briefly a pencil. The point itself is the *center* of the pencil. A finite number of concurrent lines is also called a pencil. If the center of the pencil is at infinity the pencil becomes a *parallel* pencil.

The range and pencil are linear and one-dimensional in their respective elements of point and line. They are therefore described as *linear or primitive one-dimensional forms*.

To project a range u from a point A we first form a perspection from the point, *i. e.*,

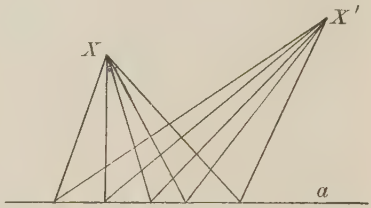
To project a pencil X we first take a section by a line a, the axis of projection. We

construct the pencil with center A. We then cut across the pencil by a line u'.



The ranges u and u' are said to be in *perspective* position or perspective ($\overline{\overline{\wedge}}$) from A. That is, *two ranges are perspective from a point A if they are in* (1, 1) *correspondence and lines joining corresponding points meet at A*. The point A is termed the *center* of the *perspectivity*. Thus perspective ranges are sections of the same pencil.

then form a perspection from a point X'. Thus section followed by perspection is projection.



The pencils X and X' are said to be perspective from the line a. That is, *two pencils are perspective from a line a if they are in* (1, 1) *correspondence and all intersections of corresponding lines lie on a*. The line a is termed the *axis of perspectivity*. Thus perspective pencils are perspections of the same range.

A pencil and a range are perspective if they are in (1, 1) *correspondence and so situated that corresponding point and line are incident.*

If now the range u' is projected from a second point A' onto a line u'' we obtain a third range u'' perspective to u' but not to u. However since u'' was derived from u we need a term to describe the relationship. They are called *projective* ($\overline{\wedge}$).

More generally any two primitive one-dimensional forms are projective if they are connected by a series of perspectivities. Symbolically, if $F \overline{\overline{\wedge}} F' \overline{\overline{\wedge}} F'' \overline{\overline{\wedge}} \ . \ . \ . \ \overline{\overline{\wedge}} F^r$, then $F \overline{\wedge} F^r$. It is to be emphasized that F and F^r may represent (*a*) two ranges, (*b*) two pencils, (*c*) a range and a pencil. In particular they might be the same form, *i. e.*, a range or a pencil may be projective with itself.

We have as immediate consequences of this definition the following useful theorems

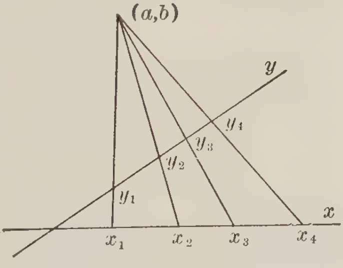
1°. If $F \overline{\overline{\wedge}} F'$ then $F \overline{\wedge} F'$.

2°. If $F \overline{\overline{\wedge}} F'$ and $F' \overline{\overline{\wedge}} F''$ then $F \overline{\wedge} F''$.

3°. If $F \overline{\wedge} F'$ and $F' \overline{\wedge} F''$ then $F \overline{\wedge} F''$, or generally If $F \overline{\wedge} F' \overline{\wedge} F'' \overline{\wedge} \ . \ . \ . \ \overline{\wedge} F^r$ then $F \overline{\wedge} F^r$.

A projective correspondence between two forms of the same kind is called a *projectivity*.

41. Double ratio.—Consider now a pencil of four lines. (See figure.) Any line x cuts the pencil in a range of four points say X_i[1] and a line y likewise determines a range of points Y_i. Take x and y as coördinate axes and denote by x_i the x-coördinates of the points X_i and by y_i the y-coördinates of the points Y_i.



[1] X_i, $i = 1, 2, 3, 4$ is a shorthand method of writing four points with the subscripts 1, 2, 3, 4.

Let the coördinates of the center of the pencil be (a, b). Then since the lines X_iY_i are on (a, b) we have

$$a/x_i + b/y_i = 1, \tag{1}$$

whence by subtraction of any pair of equations (1),

$$\frac{a(x_i - x_j)}{x_ix_j} = \frac{-b(y_i - y_j)}{y_iy_j} \tag{2}$$

hence

$$\frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_4)(x_3 - x_2)} = \frac{(y_1 - y_2)(y_3 - y_4)}{(y_1 - y_4)(y_3 - y_2)}. \tag{3}$$

Now the range y is a projection of the range x. And since the function of the x's goes over into the very same function of the y's this function defines a property inherent in the four points themselves which is unaltered by projection.[1] This number projectively attached to the four points X, in the order written, we call the *double ratio*[2] of the points and denote it variously by $(x_1x_3|x_2x_4,)$ $(X_1X_3|X_2X_4)$ or $(13|24)$.

We have at once the theorem: *If one range is projected into another the double ratio of any four points of the one is equal to the double ratio of the four corresponding points of the other.*

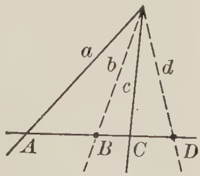
Since all lines cut a pencil of four lines in ranges having the same double ratio this double ratio may be considered characteristic of the pencil. Accordingly *we define the*

[1] Note that (3) is independent of a and b so that the equation holds when we project from any point whatever.

[2] $x_1 - x_2$ represents the directed segment $\overrightarrow{X_2X_1}$. Hence $\frac{x_1 - x_2}{x_1 - x_4}$ and $\frac{x_3 - x_2}{x_3 - x_4}$ are the ratios into which the points 1 and 3 divide the segment $\overline{24}$. It is the ratio of these ratios that is called the double ratio. The double ratio is equally the ratio of the ratios in which the points 2 and 4 divide the segment $\overline{13}$. The double ratio is thus symmetrical in the two pairs 1, 3 and 2, 4. Alternative names in common use for double ratio are *anharmonic* ratio and *cross* ratio.

double ratio of a pencil to be identical with that of any line section.

Likewise two points B, D and two lines a, c are said to have a double ratio $(ac|BD)$ defined by the identities



$$(ac|BD) \equiv (ac|bd) \equiv (AC|BD).$$

Thus double ratio is self-dual and all the properties of the double ratio of four points have their analogues in the double ratio of four lines or of two points and two lines.

42. The group of double ratios.—The value of the double ratio of four points evidently depends upon the *order* of the points. There are twenty four double ratios corresponding to the permutations of the points but not all are distinct. The number of distinct double ratios as well as their values can be found as follows. The points can be paired in three ways giving rise to the three functions $(x_1 - x_2)(x_3 - x_4)$, $(x_1 - x_3)(x_4 - x_2)$, $(x_1 - x_4)(x_2 - x_3)$. Denoting these by P, Q, and R respectively we have the useful identity associated with any four points $P + Q + R \equiv 0$. Let r be the value of the standard double ratio of four points in the natural order (1, 2, 3, 4) thus

$$(x_1x_3|x_2x_4) \equiv \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_4)(x_3 - x_2)} \equiv r \equiv -\frac{P}{R}. \qquad (1)$$

Then

1°. The interchange of two points and the simultaneous interchange of the other two leaves r unaltered. Thus $(13|24) = (24|13) = (31|42) = (42|31)$.

2°. The interchange of alternate points changes r into $1/r$. For example $(13|42) = 1/(13|24)$.

3°. The interchange of means or extremes changes r into $1 - r$. Thus $(12|34) \equiv -Q/R = (P + R)/R = 1 + P/R = 1 - r$.

Hence of the twenty four double ratios six are distinct, *viz.*,

$$\begin{array}{cccccccc} r & \dfrac{1}{1-r} & \dfrac{r-1}{r} & \text{or} & -\dfrac{P}{R} & -\dfrac{R}{Q} & -\dfrac{Q}{P} & \\ \dfrac{1}{r} & 1-r & \dfrac{r}{r-1} & & -\dfrac{R}{P} & -\dfrac{Q}{R} & -\dfrac{P}{Q} & \end{array} \quad (2)$$

These six numbers have an interesting property. If in any one of them we replace r by any double ratio of the set we recover some member of the set. The six double ratios thus form a closed set, in other words they constitute a *group*.

43. Special values of the double ratio. Harmonic sets.—In the foregoing we have supposed that all the points were distinct. If now two (and only two) of them coincide, as 3 with 4, we have $\overline{34} = 0$ and $(13|24) \equiv r = 0$. The other double ratios become 1 and ∞. Hence *if two of the four points coincide the double ratios reduce to three equal pairs with the values* 1, 0 *and* ∞.

Conversely *if the double ratio of four points is* 1, 0 *or* ∞ *two of the points coincide.* It will suffice to prove this for the standard double ratio r.

If

$$r \equiv \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_4)(x_3 - x_2)} = 0,\ x_1 = x_2 \text{ or } x_3 = x_4.$$

If

$$r = \infty, \qquad x_1 = x_4 \qquad \text{or} \qquad x_2 = x_3.$$

If

$$r = 1,\ 1 - r \equiv (12|34) = 0,\ i.e.\ (x_1 - x_3)(x_2 - x_4) = 0$$

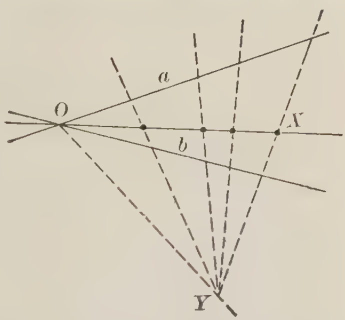
and $x_1 = x_3$ or $x_2 = x_4$. Q.E.D.

Two of the double ratios may coincide however and the four points remain distinct. For example if $r = 1/r$, $r^2 = 1$. If now $r = -1$ the points are in general distinct.

This case is particularly important as the sequel will prove. The four points in question are *harmonic* or *apolar*. In any double ratio alternate points are paired in a symmetrical way. When the points are harmonic each pair is symmetrical within itself. If x, y and a, b are two such *harmonic pairs*

$$(xy|ab) \equiv \frac{(x-a)(y-b)}{(x-b)(y-a)} = -1. \tag{1}$$

It follows directly (as well as from 1° and 2°, §42) that the double ratio is unaltered when the two pairs are interchanged or when the points of either pair are interchanged. Either point of one pair is the *harmonic conjugate* or *polar* of the other with respect to the other pair. The polar of a point with respect to a pair of points is unique but if three points are given it is possible to select a fourth in three ways to form with the others a harmonic set since 4 may be the polar of any one of the three points with respect to the other two.



Dually the polar of a line with respect to a pair of lines in the same pencil is defined to be the harmonic conjugate of the line with respect to the pair.

Observe now that the harmonic conjugate of a point with respect to a pair of lines is not unique. For if lines a and b, meeting at O, and points X, Y are harmonic pairs and Y is fixed X is free to move along OX, the harmonic conjugate of OY as to a, b. Accordingly *we define the polar line of a point Y, with respect to two lines as the locus of points X which are harmonically separated from Y by the*

line pair. It follows that OX is the polar of any point on OY with respect to a, b and vice versa.

Dually *the polar point of a line* u *with respect to a pair of points is the locus of lines* v *harmonically separated from* u *by the two points.*

Suppose now that in (1) a, b are fixed and x, y are variable points. If $x = a$ the numerator vanishes and in order that (1) shall hold the denominator must vanish at the same time. Hence

1°. If x falls at a so also does y. Similarly if x coincides with b, y coincides with b. Thus if two points of a harmonic set coincide three do.

Next write (1) in the form $\dfrac{(1-a/x)(y-b)}{(1-b/x)(y-a)} = -1.$

If $x = \infty$, $y = \dfrac{a+b}{2}$, the mid point of ab. Thus

2°. Metrically speaking, *we may define* the *center* of a segment as the polar point of the line at infinity with respect to the end points.

EXERCISES

1. Show that in two perspective ranges the point at infinity in either corresponds in general to a finite point in the other.

2. Determine the relation between corresponding segments of two perspective ranges whose center of perspectivity is at infinity. Construct the ranges.

3. Determine the relation between corresponding segments of two perspective ranges when the center of perspectivity is a finite point and the infinite points are corresponding.

4. Are two perspective ranges whose infinite points are correlative necessarily parallel?

5. Show that in two perspective pencils there is always one pair of corresponding lines which are parallel.

6. Construct two perspective pencils one of which is a parallel pencil.

7. Construct two perspective pencils whose axis of perspectivity is at infinity.

8. Construct two pencils each perspective to a third but not perspective to each other.

9. Show that three points on a line can be projected into any three points on a second line (in any order) by not more than two projections.

10. If three points x_1, x_2, x_3 and a double ratio r are given show that there is a unique point x such that $(x_1x_3|x_2x) = r$.

11. Write the twenty-four double ratios in four columns and six rows such that the double ratios in each row are equal.

12. Show that the interchange of the first two or the last two of four points changes r into $r/(r - 1)$.

13. Find the different values of the double ratio of four harmonic points.

14. If $r = -\omega$, show that the six double ratios reduce by threes to $-\omega$ and $-\omega^2$, where ω is a complex cube root of unity. The four points are then called *equianharmonic.*

15. Express the six double ratios in terms of the six trigonometric functions given $(13|24) = r = -\tan^2\dfrac{\theta}{2}$, where θ is the angle of intersection of the circles described on 13 and 24 as diameters.

16. If $(x_1x_3|x_2x_4) = -1$, verify that $\dfrac{2}{x_1 - x_3} = \dfrac{1}{x_1 - x_2} + \dfrac{1}{x_1 - x_4}$.

17. Find the polar of each of the points in a set with respect to the remaining pair in the sets $1, \omega, \omega^2$; $0, 1\ \infty$; $1, 2, 3$; $0, 1, -1$.

18. Show that if x, y and $0, \infty$ are harmonic pairs, $x = -y$.

44. Quadrangles and polygons.

A *simple n-point* is a set of n points (vertices) taken in a *definite order*, together with the n lines (sides) joining them in the chosen order. The figure is self-dual.

A *complete n-point* is a set of n points together with the $n(n - 1)/2$ lines joining them in pairs.

A *simple n-line* is a set of n lines (sides) taken in order together with the n points (vertices) of intersection of successive lines. The figure is self-dual.

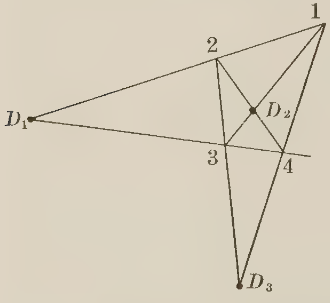
A *complete n-line* is a set of n lines together with their $n(n - 1)/2$ points of intersection.

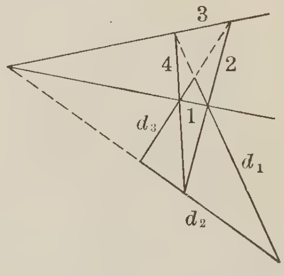
A complete 4-point or quadrangle consists of four points and their six junctions. A complete quadrangle whose vertices are 1, 2, 3, 4 contains the three simple quadrangles 1234, 1342, 1423.

Opposite sides of a quadrangle, simple or complete, are those which have no vertex in common. The intersections of opposite sides are *diagonal points.* Thus a simple quadrangle has two diagonal points, a complete quadrangle three, which form a *diagonal 3-point.*

A complete 4-line or quadrilateral consists of four lines and their six intersections. A complete quadrilateral with sides 1, 2, 3, 4 contains the three simple quadrilaterals 1234, 1342, 1423.

Opposite vertices of a quadrilateral are those which have no side in common. The junctions of opposite vertices are *diagonal lines.* A simple quadrilateral has two, a complete quadrilateral three diagonal lines which form a *diagonal 3-line.*





45. Harmonic properties.—We shall now take up a characteristic property of these figures, restricting the discussion however to quadrangles and leaving the dual as an exercise to the student. The student should assiduously cultivate this habit of dualizing for it is one of the most

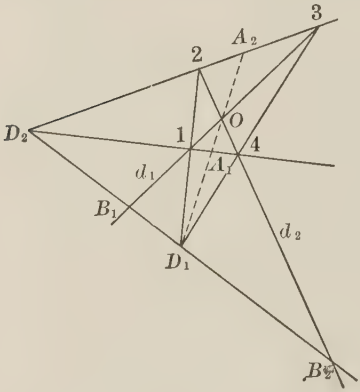
important methods in geometry. He cannot regard himself master of it until he can think through a problem in lines as well as in points and vice versa. Practice has already been afforded in making dual statements, the chief difficulty in which is due to unfortunate imperfections in terminology. Dual constructions are greatly facilitated by a judicious choice in the notation. Methods which have proved effective are to designate dual elements by (1) the same character, particularly figure, (2) the same letter, one large and one small, (3) the same character, primed and unprimed, (4) corresponding Greek and Roman letters. Many constructions are rendered almost automatic by consistent use of a good notation. One difference will be observed in actual practice. When two lines are drawn their point of intersection is incidentally constructed. On the other hand when two points are drawn in the dual figure it is necessary to make a separate construction for their junction.

We illustrate a dual notation which the student should attach to a pair of figures.

Quadrangle		*Quadrilateral*	
vertices 1 2 3 4		sides 1 2 3 4	
opposite sides	diagonal points	opposite vertices	diagonal lines
12 34	D_1	12 34	d_1
13 42	D_2	13 42	d_2
14 23	D_3	14 23	d_3

Consider now a simple quadrangle with vertices 1234, diagonal points D_1, D_2 and diagonal lines d_1, d_2, the last meeting at O. First project from D_1 the range 14 on the range 23, A_1 and A_2 being corresponding points collinear with O. Then using O as a center project 23 back onto 14

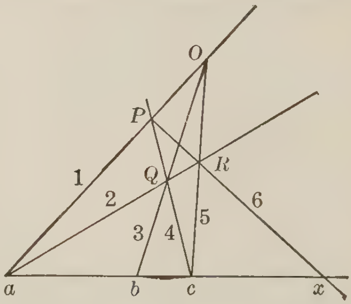
and thence onto D_1D_2. Hence, the ranges being projective, $r = (14|A_1D_2) = (23|A_2D_2) = (41|A_1D_2) = 1/r = (B_2B_1|D_1D_2)$. Since $r = 1/r$, $r = \pm 1$. But if $r = 1$ two of the points would coincide (§43). Therefore $r = -1$ and each range is made up of four harmonic points. From the last double ratio it appears that the points D_1, D_2 and the lines d_1, d_2 are harmonic pairs. Hence *in a simple quadrangle the diagonal points are harmonically separated by the diagonal lines.*



By considering in turn the simple four-points contained in the complete four-point we have the more general theorem:

Any pair of diagonal points of a complete quadrangle are harmonically separated by the pair of sides not passing through them.

46. The preceding theorem furnishes a **method of constructing with ruler only the harmonic conjugate** of a point with respect to a pair of points. Thus to find the polar point of b with respect to a, c, from a draw any two lines 1, 2 and from b any line 3 cutting 1 and 2 in O and Q. Join c to O and Q meeting 1 and 2 in P and R. Then 6 the junction of P and R, cuts line abc in the required point x. For

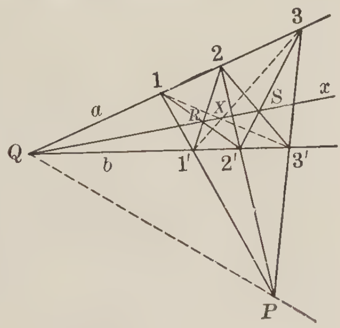


lines 1, 2 and 4, 5 form a simple quadrilateral of which a, c are diagonal points and 3, 6 diagonal lines. Hence a, c and 3, 6 or what comes to the same thing a, c and b, x are harmonic pairs. Inasmuch as the polar of a point with respect to a pair is unique this construction yields the same point regardless of the choice of 1, 2 and 3.

In particular, given a segment ac and its center b, to draw a line through a given point P parallel to ac. Connect P with a and c. Draw any line 2 through a. Then draw 3, 5 and 6 in order. 6 is parallel to ac. Why? We have thus a projective solution of a metrical problem.

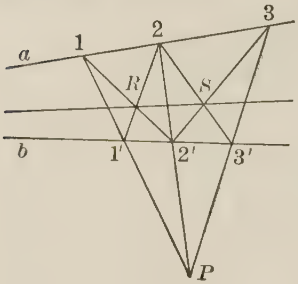
This construction can also be used to draw the harmonic conjugate of a line with respect to a pair of concurrent lines. If the polar of 3 with respect to 1, 5 is required, cut across the pencil by any line abc. Then complete the construction for the point x as above. The line Ox is the line sought.

The *polar of a point with respect to a pair of lines* can be constructed by an application of the following



Theorem.—If from any point P three, or any number of lines, are drawn cutting two lines a and b in pairs of points 1, 1′; 2, 2′; 3, 3′ etc., the pairs of cross lines like 12′, 1′ 2 etc., meet in points collinear with Q the intersection of a and b. For if R and S are two such points the quadrangles 11′ 2′ 2 and 22′ 3′ 3 have R and S respectively for diagonal points with P and Q as common diagonal points. Hence QR and QS are both harmonic conjugates of QP with respect to a, b. Therefore $QR \equiv QS$, *i. e.*, Q, R, S, are collinear. So for any

other point X like R and S. The locus of X for variable line $11'$ of the pencil P is the polar of P as to a and b.



The polar construction furnishes a *method of drawing through a point a line which shall pass through the inaccessible intersection of two lines.* If a, b are the lines and R the point, draw two lines $12'$ and $1'2$ through R determining the quadrangle $11'2'2$, with a second diagonal point P. Complete the construction as above. Then RS, the polar of P with respect to a and b, will pass through the intersection of the two lines.

EXERCISES

1. Show that a complete n-point contains $n!$ simple n-points of which $(n-1)!/2$ are distinct. How many simple hexagons in a complete hexagon? Draw a complete 5-point and pick out all the simple 5-points.

2. Construct the complete 4-point and 4-line determined by a simple quadrangle. Show that the diagonal 3-point of the 4-point is distinct from the diagonal 3-line of the 4-line. Construct the simple quadrilaterals contained in the complete 4-line and locate their diagonal points and lines. Locate the diagonal points and lines of the simple quadrangles, §44.

3. Given a triangle and its centroid (intersection of the medians). Find the diagonal 3-point of the figure considered as a 4-point.

4. If a triangle and the line at infinity are considered a 4-line draw the diagonal 3-line.

5. Dualize the statement (§45) of the harmonic properties of the complete quadrangle.

6. Pick out all the harmonic ranges and pencils you can associated with a complete quadrangle.

7. Dualize the construction (§46) for the polar of a point as to two points.

8. If a line bisects the interior angle between two lines show that its harmonic conjugate bisects the exterior angle.

9. Given a line parallel to a segment construct the center of the segment.

10. Through a point draw a line parallel to two given parallel lines.

11. Give a projective definition of a median of a triangle. Translate into a projective theorem the theorem that the medians meet in a point.

12. State the dual of the theorem of §46. Construct the polar point of a line with respect to a pair of points by dualizing the construction there given.

13. Select the metrical exercises in this list.

CHAPTER V

PROJECTIVE COÖRDINATES

47. Projective coördinates in one dimension.—We saw (§27) that the ordinary coördinate system of points on a line was completely determined when the numbers 0, 1 and ∞ were assigned to the origin, the unit point and the point at infinity. Similarly any coördinate system in one dimension is fully established as soon as a (1, 1) correspondence is set up between the number system and the elements of the domain in question, for then each element is named by a number and each number is the name of an element.

If r_0, r_1, r_∞ are the Cartesian metric coördinates of three fixed points, called the base points, on a line and x that of a variable point we define the *projective coördinate* x' of the point x to be the double ratio

$$x' = (xr_1|r_0r_\infty). \tag{1}$$

For every point x there is a unique number x' and conversely. In other words we have effected a (1, 1) correspondence between the numbers x' and the points of the line which establishes a coördinate system. When x falls at r_0, r_1, r_∞ respectively $x' = 0, 1, \infty$, *i. e., the projective coördinates of the base points are* 0, 1, ∞. The old and new names of the points are exhibited in the figure.

Observe that Cartesian coördinates are a special case of projective. For if we select 0 and 1 as base points and let

r_∞ go off to infinity where the numbers have their usual metric interpretation of distance, the Cartesian coördinate of any point x may be defined by the double ratio $(x1|0\infty) \equiv x$.

Projective coördinates may be made homogeneous exactly as metrical coördinates. Thus (1) can be written in the form

$$x' = \frac{ax + b}{cx + d} \tag{2}$$

where $a = r_1 - r_\infty$, $b = r_0(r_\infty - r_1)$, $c = (r_1 - r_0)$, $d = r_\infty(r_0 - r_1)$. Then replacing x by x_1/x_2 and x' by x_1'/x_2' we have

$$\frac{x_1'}{x_2'} = \frac{ax_1 + bx_2}{cx_1 + dx_2} \tag{3}$$

which says that the homogeneous projective coördinates x_1', x_2' of the point x are proportional to $ax_1 + bx_2$ and $cx_1 + dx_2$ where x_1 and x_2 are the homogeneous Cartesian coördinates of x. Equation (3) may be made non-homogeneous in x or x' or both at pleasure, reducing in the last case to the original form (2).

Dually to establish a coördinate system for a pencil of lines we select three lines of the pencil at random with coördinates say s_0, s_1, s_∞. Then the projective coördinate u' of any line u is defined by the double ratio

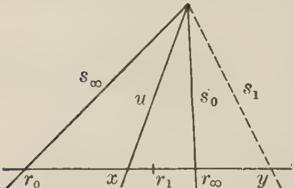
$$u' = (us_1|s_0s_\infty) \equiv \frac{Au + B}{Cu + D}. \tag{4}$$

As before the projective coördinates of the base lines are 0, 1 and ∞. Likewise the homogeneous projective coördinates of the line are

$$\sigma u_1' = Au_1 + Bu_2, \quad \sigma u_2' = Cu_1 + Du_2 \tag{5}$$

where σ is a factor of proportionality and u_1, u_2 are the homogeneous coördinates of the line in the old system.

But it is desirable to have a framework to which may be referred alike the points of a range and the lines of a pencil.



For this purpose we need three points of the range and three lines of the pencil. Two of each are called the reference elements, the remaining are the unit point and line. Let the reference points r_0, r_∞ and the reference lines s_∞, s_0 respectively be incident. And further let the unit point r_1 be the polar of the unit line s_1 as to the reference points (or dually). Then the projective coördinates of any point x and any line u are

$$x' = (xr_1|r_0r_\infty), \qquad u' = (us_1|s_0s_\infty). \tag{6}$$

If y is the point of the range cut out by s_1 we have from the harmonic property

$$-1 = (r_1y|r_0r_\infty) = \frac{(r_1 - r_0)(y - r_\infty)}{(r_1 - r_\infty)(y - r_0)}. \tag{7}$$

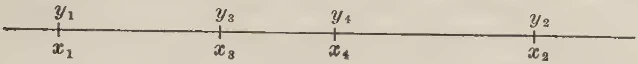
Then the condition that u pass through x is

$$\begin{aligned} u' = (xy|r_\infty r_0) &= \frac{(x - r_\infty)(y - r_0)}{(x - r_0)(y - r_\infty)} \\ &= -\frac{(x - r_\infty)(r_1 - r_0)}{(x - r_0)(r_1 - r_\infty)} \text{ (by (7))} \\ &= -(xr_1|r_\infty r_0) = -1/x'. \end{aligned} \tag{8}$$

Hence the incidence condition of point x' *and line* u' *is* $u'x' + 1 = 0$, *or in homogeneous coördinates* $u_1'x_1' + u_2'x_2' = 0$.

48. Transformation of coördinates.—Equation (2) §47 may be looked upon as a *renaming* of the points of a range for the old metrical name x of any point is replaced by the new projective name x'. More generally let us ask how the coördinates of a point may be transformed from any projective coördinate system to another. Let x_i be the coör-

dinates of the points referred to the old system and y_i the coördinates of the corresponding points in the new. The first requirement is that there shall be a (1, 1) correspondence between the numbers x and y. We next assume that the



transformation is *continuous, i. e.*, as x approaches the value x_i as a limit in the one system y approaches y_i as a limit in the other. It follows that the relation between x and y must be linear in each. The most general *bi-linear* or *lineo-linear* equation satisfying both conditions is

$$cxy - ax + dy - b = 0 \text{ or } y = \frac{ax + b}{cx + d}, \tag{1}$$

where the coefficients are arbitrary constants subject only to the condition $ad - bc \neq 0$. This restriction amounts to saying that if two names are distinct in the old system they will be distinct in the new. For if x_i, x_j and y_i, y_j are two pairs of corresponding coördinates

$$y_i - y_j = \frac{(ad - bc)(x_i - x_j)}{(cx_i + d)(cx_j + d)}. \tag{2}$$

Hence if $x_i \neq x_j$ and $ad - bc \neq 0$, then $y_i \neq y_j$.

By means of equation (2) it is easily verified that $(y_1y_3|y_2y_4) = (x_1x_3|x_2x_4)$. *Hence the double ratio of the four points is independent of the coördinate system by which the points are named.*

49. Alias and alibi.—The bilinear equation, which in the abstract we may call a *linear transformation*, is subject to a second interpretation quite as important as the other. In the previous paragraph we thought of x and y as coördinates of the same point in different coördinate systems, when (1) was the process which changed the names of the points leaving the points themselves fixed in position.

We may however regard x and y as coördinates of corresponding points in different ranges and in particular as cöordinates (in the same system) of different points in the same range.[1]) Then (1) becomes the operation which picks up the point x bodily and sends it into the point y in a one-to-one way, *in other words it is the analytic definition of a projection or projectivity* (§40). There are thus two aspects of a linear transformation in one dimension. Professor Morley has happily called the first of these, which is a change of name, an *alias* and the second, which is a change of place, an *alibi*. The standard term for an alibi whether for points or lines is *collineation.*

50. We shall now develop **some properties of the linear transformation** as applied to projective ranges situated on different lines. But the analytic proofs imply analogous theorems in the transformation of coördinates or collineation on the same line together with the duals. Because of their importance we shall first recapitulate formally the results of §§48–49 (theorems 1° and 2°).

1°. *If two ranges are projective then corresponding points x, y are connected by a relation of the form*

$$y = \frac{ax + b}{cx + d}$$

2°. *Conversely if x and y satisfy a bilinear equation they may be considered corresponding points in two projective ranges.* For the correspondence is one-to-one and the double ratio of four points x is equal to the double ratio of the four corresponding points y.

Cor. *If two ranges are in* (1, 1) *correspondence they are projective.* For corresponding points are connected by a relation

$$y = \frac{ax + b}{cx + d}$$

[1] Each interpretation of the linear transformation has of course a dual.

The general linear transformation represents a three-parameter family since it contains three essential constants. This means that three independent conditions are necessary to specify it. In particular any three points x of the one range may be sent into any three points y of the other,[1] *i. e.*, three pairs of values may be assigned arbitrarily to x and y giving rise to three linear equations which are sufficient to determine the ratios $a : b : c : d$. Thus suppose that when $x = 0, -1, 2$, $y = -1, 0, 3$ respectively. We must have $b + d = 0$, $a - b = 0$, $2a + b - 6c - 3d = 0$, whence $a = b = c = -d$ and the transformation is $y = (x + 1)/(x - 1)$. Obviously now when either variable is given the other is determined. We may say then

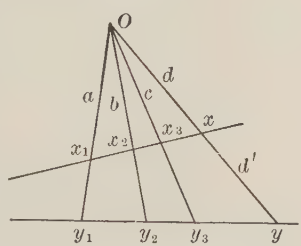
3°. *A projective correspondence between two linear one-dimensional forms is fully established when three pairs of corresponding elements are given.*

If x_1, y_1; x_2, y_2; x_3, y_3 are the three fixed pairs of corresponding points the transformation can be expressed in terms of them in virtue of the equality of double ratios thus

$$\frac{(y - y_1)(y_2 - y_3)}{(y - y_3)(y_2 - y_1)} = \frac{(x - x_1)(x_2 - x_3)}{(x - x_3)(x_2 - x_1)}$$

where x, y are a variable pair of corresponding points.

Ex. (Cor.) *If three elements of one form coincide with the three corresponding elements of the other the forms coincide throughout.*

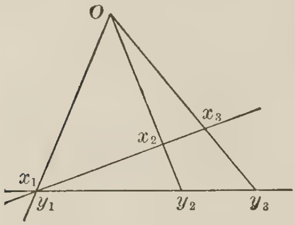


4°. *If the junctions of three pairs of corresponding points of two projective ranges are concurrent the ranges are perspective.*

Let x_1y_1, x_2y_2, x_3y_3, denoted by a, b, c, meet at O and let x and y be any other pair of corresponding points. Calling

[1] Cf. Ex. 9, §43.

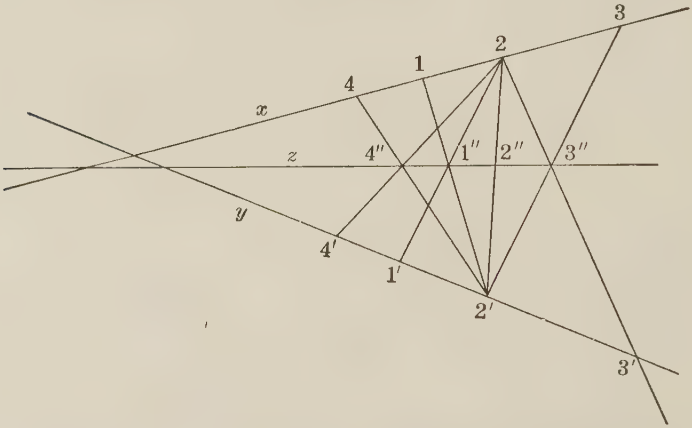
Ox and Oy d and d' respectively we must have $(ac \mid bd) = (x_1x_3 \mid x_2x) = (y_1y_3 \mid y_2y) = (ac \mid bd')$. Hence $d \equiv d'$, *i. e.*, x and y are perspective from O.



COR. *If two projective ranges, lying on distinct lines, have a pair of corresponding points in coincidence the ranges are perspective.*

For any two pairs of corresponding points as x_2, y_2 and x_3, y_3 determine a point O which may be joined to $x_1 \equiv y_1$. We have thus three pairs of corresponding points perspective from O, a case of the theorem.

5°. *If two ranges are projective there exists a third range perspective to both.*



Let points 1, 2, 3 of range x correspond respectively to points 1′, 2′, 3′ of range $y \overline{\wedge} x$. Now form a perspection of y from (say) 2 and a perspection of x from 2′. If the first pencil is denoted by X and the second by Y we have

$$X \overset{=}{\wedge} y \overline{\wedge} x \overset{=}{\wedge} Y. \quad \text{Hence } X \overline{\wedge} Y.$$

If corresponding lines of these two pencils are taken as $12'$ and $1'2$ etc. then $22'$ is a self-corresponding line. Therefore $X \overline{\overline{\wedge}} Y$ (4°, COR. (dual)). The axis of perspectivity $z (\equiv 1''\, 2''\, 3'')$ is perspective to both x and y. Q. E. D.

This theorem enables us, given any three pairs of corresponding points of two projective ranges, to construct the ranges.

Let 4 be any fourth point of x (see figure). Joining 4 to $2'$ we determine $4''$ on z. Then $24''$ cuts y in $4'$, the partner of 4.

EXERCISES

1. Show that if $ad = bc$ in the linear transformation all points x correspond to a single point y.

2. Determine the transformation that sends (*a*) 0, 1, ∞ into 1, ∞, 0; (*b*) 1, 2, 3 into 2, 1, 3, (*c*) 0, i, 1 into -1, ∞, i, (*d*) 0, 1, ∞ into ω^2, 1, ω.

3. Find the effect of the transformation $y = \dfrac{x+2}{-x+1}$ on the cubic $2y^3 - 3y^2 + 15y - 7 = 0$. Interpret in two ways.

4. Apply the homogeneous transformations $t_1 = t_2' - \omega t_1'$, $t_2 = t_1' - \omega t_2'$ to the equation $t_1^4 t_2 - 2t_1^3 t_2^2 + 2t_1^2 t_2^3 - t_1 t_2^4 = 0$.

5. Find a transformation that sends $y_1^2 - 5y_1y_2 + 6y_2^2 = 0$ into $x_1x_2 = 0$.

6. Find the transformation that sends $y^3 + 1 = 0$ into $x^3 - 1 = 0$.

7. Dualize theorem 5° and the construction.

8. If x and y (Th. 5°) meet at O, find the partner of O considered (*a*) as a point of x, (*b*) as a point of y. Thence show that the construction for z is independent of the choice of the centers X and Y. Dualize.

9. Find the point on each range corresponding to the point at infinity on the other.

10. Construct z given that 3 and $3'$ are the points at infinity on the two ranges.

11. Can x, y, z be concurrent?

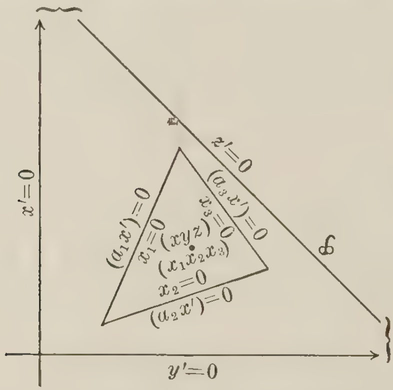
51. Projective coördinates in two dimensions.—Projective coördinates of points and lines in a plane may be defined as in one dimension by double ratios. We shall

however adopt a method that leads directly to forms suggested by the final results of §47, where we saw that the homogeneous projective coördinates are proportional to linear combinations of homogeneous metrical coördinates.

For this purpose we select three non-concurrent lines, called the *triangle of reference*, the equations of whose sides in homogeneous Cartesian coördinates are

$$a_ix' + b_iy' + c_iz' \equiv (a_ix') = 0, \qquad i = 1, 2, 3. \quad (1)$$

If now x, y, z are the homogeneous Cartesian coördinates of a point, we define the homogeneous projective coördinates



x_1, x_2, x_3 of the point as numbers proportional to expressions obtained by substituting x, y, z in the (Cartesian) equations of the sides of the triangle of reference thus

$$\rho x_i = a_ix + b_iy + c_iz, \qquad i = 1, 2, 3, \quad (2)$$

where ρ is an arbitrary factor of proportionality. As in the old system of homogeneous coördinates only the ratios of the numbers x_i are of consequence.

The selection of the reference lines however is not sufficient to establish a system of projective coördinates. For the values on the right in equations (2) depend on the form

in which the equations of the lines are taken. If the equations are multiplied respectively by arbitrary constants k_1, k_2, k_3, which in no way affects the lines themselves, we shall obtain the more general definitions of projective coördinates

$$\begin{aligned} \rho x_1 &= k_1(a_1x + b_1y + c_1z) \\ \rho x_2 &= k_2(a_2x + b_2y + c_2z) \\ \rho x_3 &= k_3(a_3x + b_3y + c_3z). \end{aligned} \tag{3}$$

The k's may be fixed in any convenient manner. In particular they may be so chosen that any point (x, y, z), which indeed does not lie on a side of the fundamental triangle, will have its projective coördinates all equal. This is done by substituting (x, y, z) in (3) and asking that $k_1x_1 = k_2x_2 = k_3x_3$. The choice of this point, called the *unit point* since its projective coördinates are proportional to 1, 1, 1, completes the determination of the coördinate system. It remains to decide on a suitable point for the unit point. We shall see below how it may be picked to the best advantage.

The determinant of equations (3)

$$d = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

cannot vanish since the lines are not on a point.

The equations of the sides of the triangle of reference in projective coördinates are $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, for when (x, y, z) lies on the side $(a_ix') = 0$ we must have $(a_ix) = 0$. It follows that the projective coördinates of a point cannot all be zero else the sides of the triangle of reference would all pass through the point.

If a geometric meaning of the new coördinates is desired it will be recalled that the distances of the point (x, y, z) from the sides of the reference triangle are

$$d_i = \frac{a_ix + b_iy + c_iz}{z\sqrt{a_i^2 + b_i^2}}, \tag{4}$$

so that we may write from (3) $\rho x_i = m_i d_i$ where

$$m_i = k_i z\sqrt{a_i^2 + b_i^2}.$$

Hence the projective coördinates of a point are proportional to fixed multiples of the distances of the point from the sides of the triangle of reference.

52. Projective line coördinates.—To establish a system of projective line coördinates we shall need dually three non-collinear points and a unit line. For the points we use the vertices of the triangle of §51. Their equations in homogeneous Plücker coördinates are (§28)

$$A_i u' + B_i v' + C_i w' = 0, \qquad i = 1, 2, 3, \tag{1}$$

where the coefficients are the cofactors of corresponding elements in the determinant d. The determinant of equations (1) we shall denote by D thus

$$D = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

and the cofactors of elements of D will be represented by corresponding Greek letters. It is well known from the theory of determinants that $D = d^2$ a property that will be used in subsequent reductions and which the student may verify.

If K_1, K_2, K_3 are arbitrary multipliers and σ a factor of proportionality the homogeneous projective coördinates (u_1, u_2, u_3) of the line whose Plücker coördinates are u, v, w are defined by the equations

$$\begin{aligned} \sigma u_1 &= K_1(A_1 u + B_1 v + C_1 w) \\ \sigma u_2 &= K_2(A_2 u + B_2 v + C_2 w) \\ \sigma u_3 &= K_3(A_3 u + B_3 v + C_3 w). \end{aligned} \tag{2}$$

The projective coördinates of a line are thus three numbers proportional to fixed but arbitrarily chosen multiples of the expressions obtained by substituting the Plücker coördinates of the line in the equations of the reference points.

Any line not on a vertex of the triangle of reference may be taken as the line (1, 1, 1). The choice of the unit line suffices to fix the K's,—when the coördinate system is completely determined.

53. We have now to consider the **equations of curves in projective coördinates.** For the utility of any coördinate system is conditioned on the forms assumed by the equations of loci which are referred to it.

The equations ((3) §51 and (2) §52) which define projective coördinates may be regarded as effecting a renaming of the points and lines of the plane. For given the metrical coördinates of an element the projective coördinates are expressed by those equations. Again since neither d nor D can vanish the equations can be solved for x, y, z and u, v, w and the old coördinates expressed in terms of the new thus

$$\begin{aligned} x &= \frac{\rho}{d}\left(\frac{A_1}{k_1}x_1 + \frac{A_2}{k_2}x_2 + \frac{A_3}{k_3}x_3\right) \\ y &= \frac{\rho}{d}\left(\frac{B_1}{k_1}x_1 + \frac{B_2}{k_2}x_2 + \frac{B_3}{k_3}x_3\right) \\ z &= \frac{\rho}{d}\left(\frac{C_1}{k_1}x_1 + \frac{C_2}{k_2}x_2 + \frac{C_3}{k_3}x_3\right) \end{aligned} \tag{1}$$

$$\begin{aligned} u &= \frac{\sigma}{D}\left(\frac{\alpha_1}{K_1}u_1 + \frac{\alpha_2}{K_2}u_2 + \frac{\alpha_3}{K_3}u_3\right) \\ v &= \frac{\sigma}{D}\left(\frac{\beta_1}{K_1}u_1 + \frac{\beta_2}{K_2}u_2 + \frac{\beta_3}{K_3}u_3\right) \\ w &= \frac{\sigma}{D}\left(\frac{\gamma_1}{K_1}u_1 + \frac{\gamma_2}{K_2}u_2 + \frac{\gamma_3}{K_3}u_3\right). \end{aligned} \tag{2}$$

Now the coördinates in either system are linear, homogeneous functions of the coördinates in the other. Hence an equation homogeneous and of degree n in the coördinates of one system will be transformed by the appropriate substitutions into a homogeneous equation of the same degree in the coördinates of the other. It follows that

1°. *Any equation,* $f^n(x_1, x_2, x_3) = 0$, *homogeneous and of degree n in projective point coördinates represents a curve of order n.* For the equation will be transformed by the substitutions (3) §51 into a homogeneous equation of the same degree in x, y, z. *Dually an equation,* $\phi^m(u_1, u_2, u_3) = 0$, *homogeneous and of degree m in projective line coördinates represents a curve of class m.*

2°. *Conversely a curve of order n is represented by a homogeneous equation of degree n in projective point coördinates.* For the Cartesian equation of the curve under the substitutions (1) is transformed into a homogeneous equation of degree n in x_1, x_2, x_3. *Dually a curve of class m is represented by a homogeneous equation of degree m in projective line coördinates.*

54. The equation of a line in projective coördinates is thus linear and homogeneous in x_1, x_2, x_3 and u_1, u_2, u_3 but the precise form of the equation is still unknown. To find it we substitute for the letters in $ux + vy + wz = 0$ their values from (1) and (2) above, supposing for the moment that the k's are undetermined. We have thence

$$\begin{aligned}\frac{dD}{\rho\sigma}(ux + vy + wz) = {}&\\ &\left(\frac{\alpha_1}{K_1}u_1 + \frac{\alpha_2}{K_2}u_2 + \frac{\alpha_3}{K_3}u_3\right)\left(\frac{A_1}{k_1}x_1 + \frac{A_2}{k_2}x_2 + \frac{A_3}{k_3}x_3\right)\\ + &\left(\frac{\beta_1}{K_1}u_1 + \frac{\beta_2}{K_2}u_2 + \frac{\beta_3}{K_3}u_3\right)\left(\frac{B_1}{k_1}x_1 + \frac{B_2}{k_2}x_2 + \frac{B_3}{k_3}x_3\right)\\ + &\left(\frac{\gamma_1}{K_1}u_1 + \frac{\gamma_2}{K_2}u_2 + \frac{\gamma_3}{K_3}u_3\right)\left(\frac{C_1}{k_1}x_1 + \frac{C_2}{k_2}x_2 + \frac{C_3}{k_3}x_3\right), \quad (1)\end{aligned}$$

the right side of which may be rearranged in the form

$$\sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\alpha_i A_j + \beta_i B_j + \gamma_i C_j}{K_i k_j} u_i x_j. \tag{2}$$

Now it is shown in the theory of determinants or may be verified directly that

$$\alpha_i A_j + \beta_i B_j + \gamma_i C_j$$

is merely the expanded form of D when $i = j$ but identically zero when $i \neq j$. Hence of the nine terms in (2) all vanish but three and (1) reduces to

$$\frac{d}{\rho\sigma}(ux + vy + wz) = \frac{u_1 x_1}{K_1 k_1} + \frac{u_2 x_2}{K_2 k_2} + \frac{u_3 x_3}{K_3 k_3} \tag{3}$$

where the D has been divided out. But the k's are still at our disposal. Choosing the unit point and line in such a way that $K_1 k_1 = K_2 k_2 = K_3 k_3$, we have the equation of the line in projective coördinates in the elegant form

$$(ux) \equiv u_1 x_1 + u_2 x_2 + u_3 x_3 = 0. \tag{4}$$

We shall assume in the sequel when projective coördinates are employed that the k's have been so chosen that the equation of the line takes the form (4). The gain is not only one of simplicity,—the important result is that then *the projective coördinates of the line are the three coefficients in its equation.*

Dually if we hold fast the x's while the u's vary, (4) is the projective equation of the point (x_1, x_2, x_3).

Equation (4) is thus the *incidence condition* of point (x_1, x_2, x_3) and line (u_1, u_2, u_3) in projective coördinates.

Hereafter unless the contrary is explicitly stated the variables in an equation are to be interpreted as projective coördinates even when for convenience of writing we use x, y, z, etc. In actual practice however there is little occasion to differentiate between the two systems of homo-

geneous coördinates[1] for as long as we are dealing with projective properties the algebra is identical. Thus the whole of §28 together with solutions of the appended exercises and the first two paragraphs of §29 are equally valid when the variables represent projective coördinates.

EXERCISES

1. Show that the projective coördinates of a line are proportional to fixed multiples of the distances of the line from the vertices of the triangle of reference.

2. What are the projective coördinates of the sides of the triangle of reference?

3. Write the projective equations and coördinates of the vertices of the reference triangle.

4. Show that when the k's are fixed as above the unit elements are polar point and line (§28) with respect to the fundamental triangle.

In Exs. 5–13 the equations of the axes in Cartesian coördinates are $x + y + 2z = 0$, $x - y + z = 0$, $2x + 3y - z = 0$, the unit point is $(3, -4, 2)$ (in Cartesians) and the unit line is taken in the standard way.

5. Find the projective coördinates of $(1, 1, 1)$, $(0, 1, 1)$, $(-1, 0, 1)$, $(2, -3, 4)$, (x, y, z).

6. Calculate d and D and verify that $D = d^2$.

7. Find the projective coördinates of the lines whose homogeneous Plücker coördinates are $(1, 1, 1)$, $(1, -1, -1)$, $(3, 2, -5)$, $(a, -a, 0)$, (u, v, w).

8. What are the Cartesian and Plücker coördinates respectively of the vertices and sides of the reference triangle?

9. Find the Cartesian coördinates of the points whose projective coördinates are $(3, 0, -5)$, $(1, -1, 1)$, $(6, 7, 4)$, (x_1, x_2, x_3).

10. Find the Plücker coördinates of the lines whose projective coördinates are $(1, 1, 1)$, $(5, 2, 0)$, $(-1, 0, 1)$, (u_1, u_2, u_3).

11. Write the equations of the sides and vertices of the Cartesian-Plücker triangle of reference in projective coördinates. Find the projective coördinates of the sides and vertices.

12. Write the equations of the following lines and points in projective coördinates

[1] Or any two systems of homogeneous coördinates in which the equation of the line takes the form (4).

$$x + y + z = 0, \qquad u + v + w = 0,$$
$$4x - 3y + z = 0, \qquad 2u - 7v + 3w = 0,$$
$$x - y = 0 \qquad u + v = 0$$
$$ax + by + cz = 0, \qquad au + bv + cw = 0.$$

13. Write the equations of the following lines and points in Cartesian and Plücker coördinates

$$x_1 - x_2 = 0, \qquad u_1 + u_2 + u_3 = 0,$$
$$x_2 + x_3 = 0, \qquad u_3 - u_1 = 0,$$
$$ax_1 + bx_2 + cx_3 = 0, \qquad au_1 + bu_2 + cu_3 = 0.$$

14. Find the equation of the line $ux + vy + wz = 0$ in projective coördinates when the unit point and line are taken as $(3, 2, 5)$ and $(2, -3, 4)$ respectively. Are coördinates of the line the coefficients in its equation?

15. Find the equation of $b^2x^2 + a^2y^2 - a^2b^2z^2 = 0$ in projective coördinates when the projective coördinates are defined by the equations $x_1 = x - az$, $x_2 = x + az$, $x_3 = y$.

16. The coördinates of the vertices of a quadrangle are $(3, 2, 5)$, $(5, -2, 3)$, $(1, 4, -1)$, $(-3, 0, 3)$. Find the equations of the six sides and the coördinates of the vertices of the diagonal triangle.

17. The equations of the sides of a quadrilateral are $x_1 - 2x_2 - x_3 = 0$, $x_1 + 2x_3 = 0$, $3x_1 + x_2 + 2x_3 = 0$, $2x_1 - x_2 - 3x_3 = 0$. Find the equations of the six vertices and the coördinates of the sides of the diagonal triangle.

55. Double ratio in terms of a parameter.—We have expressed the double ratio of four points in terms of (*a*) the line segments joining the points and (*b*) the coördinates of the points, whether metric or projective (§§41, 48). There is another form of great importance since it enables us to write down the double ratio of the points from their equations. We shall solve the dual problem first involving as it does the more familiar ideas.

Let there be given two lines whose equations are

$$\begin{aligned} U &\equiv (ax) \equiv a_1x_1 + a_2x_2 + a_3x_3 = 0, \\ V &\equiv (bx) \equiv b_1x_1 + b_2x_2 + b_3x_3 = 0. \end{aligned} \tag{1}$$

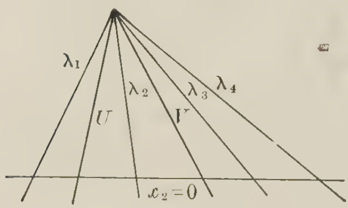
Then for every value of λ

$$(ax) + \lambda(bx) = 0 \tag{2}$$

is the equation of a line on the intersection of the two. For (*a*) the equation is of the first degree and (*b*) since the coördinates of the intersection make $(ax) = 0$ and $(bx) = 0$ separately they will also satisfy (2). For varying λ, (2) is the equation of a pencil whose base lines are (ax) and (bx). We have thus a (1, 1) correspondence between the values of λ and the lines of the pencil. The λ which is associated with a line in this correspondence is called the *parameter of the line.*

Consider now the pencil of four lines $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.[1] The equations of the lines are

$$(ax) + \lambda_i(bx), \qquad i = 1, 2, 3, 4. \tag{3}$$



Since the double ratio of the pencil is identical with that of any line section, the double ratio of the four lines is equal to the double ratio of their intercepts on any axis. Setting $x_2 = 0$ in (3) we find as the intercepts on $x_2 = 0$

$$\mathrm{x}_i \equiv \frac{x_1^{(i)}}{x_3^{(i)}} = -\frac{b_3\lambda_i + a_3}{b_1\lambda_i + a_1}. \tag{4}$$

But this equation has the form of the linear transformation ((1) §48). Hence $(\mathrm{x}_1\mathrm{x}_3|\mathrm{x}_2\mathrm{x}_4) = (\lambda_1\lambda_3|\lambda_2\lambda_4)$, *i. e., the double ratio of the four lines is equal to the double ratio of their parameters.*

In particular the parameters of U and V are 0 and ∞ respectively. Hence to find the double ratio of U and V (say λ_1 and λ_3) and any other pair of lines λ_2 and λ_4 of the pencil we seek the double ratio of

$$U = 0, \ V = 0, \ U + \lambda_2 V = 0, \ U + \lambda_4 V = 0. \tag{5}$$

[1] We shall refer to the line whose parameter is λ as the line λ.

The double ratio is thus $(0\,\infty\,|\,\lambda_2\lambda_4) = \lambda_2/\lambda_4$.[1]

If $\lambda_2 + \lambda_4 = 0$ the lines λ_2, λ_4 are harmonic with the base lines, *i. e.*,

$$U = 0,\ V = 0,\ U + \lambda V = 0,\ U - \lambda V = 0 \tag{6}$$

are lines of a harmonic pencil, a form which may be regarded as standard.

If u_1, u_2, u_3 are coördinates of a variable line of the pencil (2) we have

$$\begin{aligned} u_1 &= a_1 + \lambda b_1 \\ u_2 &= a_2 + \lambda b_2 \\ u_3 &= a_3 + \lambda b_3. \end{aligned} \tag{7}$$

These equations which express the coördinates of lines on a point (the center of the pencil) in terms of a parameter are called *parametric equations of the point.*

Here as throughout the discussion the λ can be made homogeneous, as is sometimes convenient, by setting $\lambda = \lambda_2/\lambda_1$. Equation (2) then becomes $\lambda_1(ax) + \lambda_2(bx) = 0$, equations (7) are $u_i = a_i\lambda_1 + b_i\lambda_2$; the parameters of the base lines are 1, 0 and 0, 1, etc. On the other hand since we are dealing with projective properties the lines might have been taken in the non-homogeneous or any other convenient form.

Dually given two points

$$\begin{aligned} X &\equiv (au) \equiv a_1u_1 + a_2u_2 + a_3u_3 = 0, \\ Y &\equiv (bu) \equiv b_1u_1 + b_2u_2 + b_3u_3 = 0, \end{aligned} \tag{8}$$

then *for every value of* λ

$$(au) + \lambda(bu) = 0 \tag{9}$$

is the equation of a point on the junction of the two. For

[1] The parameter of a line may be thought of quite properly as a *coördinate* of the line, one number being sufficient to determine a line of a pencil. Indeed if U and V are taken as reference lines and λ_4 as the unit line then for any fourth line λ we have $(0\,\infty\,|\,\lambda 1) = \lambda$.

varying λ, (9) represents a range of points. The double ratio of the four points λ_i whose equations are

$$(au) + \lambda_i(bu) = 0, \qquad i = 1, 2, 3, 4, \tag{10}$$

is equal to the double ratio of the parameters of the points.

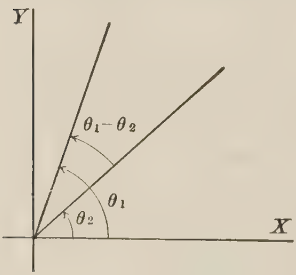
In particular the double ratio of the base points X, Y and the pair λ_2, λ_4 is λ_2/λ_4. Whence $X + \lambda Y = 0$, $X - \lambda Y = 0$ are a pair of points harmonic with $X = 0$, $Y = 0$.

The equations

$$\begin{aligned} x_1 &= b_1\lambda + a_1 \\ x_2 &= b_2\lambda + a_2 \\ x_3 &= b_3\lambda + a_3 \end{aligned} \tag{11}$$

which express the coördinates of a variable point (x_1, x_2, x_3) of the line (axis of the range) in terms of a parameter are called *parametric equations of the line.*

56. Projective-metric definition of angle.—The results of the last section enable us to express the angle between two lines as a double ratio. For taking the vertex of the angle as origin of rectangular coördinates, let θ_1 and θ_2 be the angles which the lines make with the x-axis. Then if θ is the angle between the lines we have $\theta = \theta_1 - \theta_2$. The equations of the lines may be written $y = x \tan \theta_1$, $y = x \tan \theta_2$ while the circular rays through the origin are $y = ix$, $y = -ix$. Hence the double ratio r of the four lines is



$$r = (\tan\theta_1 \quad \tan\theta_2 \mid i \quad -i) = \frac{(\tan\theta_1 - i)(\tan\theta_2 + i)}{(\tan\theta_1 + i)(\tan\theta_2 - i)}$$

$$= \frac{(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_1 - i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)} = \frac{e^{i\theta_1}e^{-i\theta_2}}{e^{-i\theta_1}e^{i\theta_2}}$$

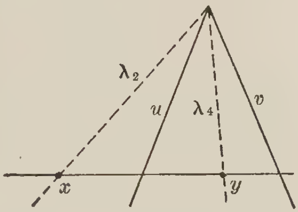
$$= e^{2i(\theta_1 - \theta_2)} = e^{2i\theta}$$

where e is the Naperian base of logarithms. Therefore

$$\log r = 2i\theta \text{ or } \theta = \frac{1}{2i} \log r.$$

In other words, *the angle between two lines is a definite multiple of the logarithm of the double ratio of the two lines and the circular points.*[1]

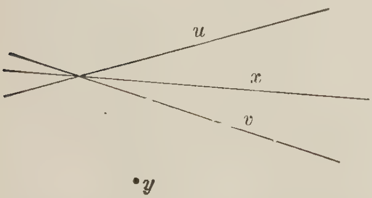
57. Double ratio of two points and two lines.—Let the equations of the lines be $(uz) = 0$, $(vz) = 0$, where z is the running coördinate, and the coördinates of the points be (x_1, x_2, x_3), (y_1, y_2, y_3). Then $(uz) + \lambda(vz) = 0$ is a variable line of the pencil determined by u and v. If this line is on x we must have $(ux) + \lambda(vx) = 0$. The corresponding value of λ, say λ_2, is $\lambda_2 = -(ux)/(vx)$. Similarly the parameter λ_4 of the line on y is $\lambda_4 = -(uy)/(vy)$. Hence *the double ratio of the two points and two lines which is λ_2/λ_4 is equal to*

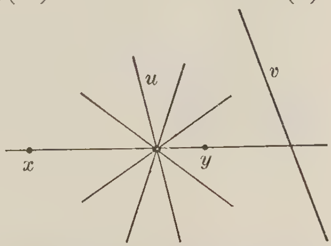


$$\frac{(ux)(vy)}{(uy)(vx)}. \tag{1}$$

Cor. 1. *The condition that the points and lines form harmonic pairs is*

$$(ux)(vy) + (uy)(vx) = 0. \tag{2}$$





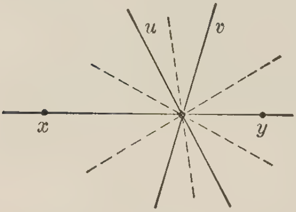
For fixed u, v, y and variable x, this represents a line, *viz.*, the polar line of y as to the line pair u, v. On the other

[1] E. Laguerre, *Nouv. Ann. Math.*, Paris, **12** (1853).

7

hand if x, y, v are fixed and u variable (2) is the equation of the polar point of v as to x, y.

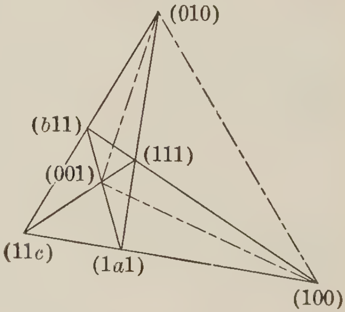
COR. 2. Ex. *If the double ratio is equal to* 1 *the intersection of the lines is on the junction of the points.* Or



$$(ux)(vy) - (uy)(vx) = 0 \quad (3)$$

for variable x is the equation of the line joining y to the intersection of u, v. While for variable u it is the equation of the point of intersection of v with the junction of x, y.

58. The projective coördinates of four points can be taken in the form 1, ±1, ±1.—For we may select the diagonal triangle of the points as reference triangle and one of the points as (1, 1, 1). Then the line joining (0, 1, 0) and (1, 1, 1) is $x_1 = x_3$. But this line passes through a second one of the four points. Accordingly we may assign to that point, two of the coördinates being equal, the numbers (1, a, 1). Similarly the coördinates of the remaining points will be (b, 1, 1) and (1, 1, c). Since (1, a, 1), (b, 1, 1) and (0, 0, 1) are on a line we must have



$$\begin{vmatrix} 1 & a & 1 \\ b & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0, \text{ or } ab = 1.$$

And from symmetry $bc = ca = ab = 1$. Whence $a = b = c = -1$, which proves the theorem.

EXERCISES

1. Examine the relation of the two points and two lines §57 when the double ratio is 0 or ∞.

2. Show that the double ratio of the x_i coördinates of four collinear points in the plane is the same as the double ratio of the points.

3. From the results of §55, 58 prove the harmonic property of a quadrangle.

4. Find the double ratio of the lines $x - 1 = 0$, $y - 2 = 0$, $x - y + 1 = 0$, $x - 3y + 5 = 0$.

5. Find the equations and the double ratio of the lines joining the point (a_1, a_2, a_3) to the four points $(1, \pm 1, \pm 1)$.

6. Show that the double ratio of the lines joining a point of the curve $x_2^2 - 4x_1x_3 = 0$ to the four points $(1, 2, 1)$, $(1, -2, 1)$, $(4, 4, 1)$ and $(0, 0, 1)$ is constant.

7. Show that I and J are harmonic with any pair of perpendicular lines, obtaining thus a new definition of perpendicularity.

8. Show that the logarithmic definition of a right angle leads to the equation $e^{i\pi} + 1 = 0$.

9. Write the parametric equations of the centers of the pencils in Exs. 4, 5.

10. Find the parametric equations of the lines on the points $(2, -5, 3)$, $(4, -3, -7)$; $u_1 + u_2 + u_3 = 0$, $3u_1 - u_2 + 6u_3 = 0$.

11. Find the double ratio of the points $2u_1 + u_2 - u_3 = 0$, $u_1 - 5u_2 + 3u_3 = 0$ and the lines $x_1 - x_2 - x_3 = 0$, $8x_1 + 5x_2 + 6x_3 = 0$.

12. Find the equation of the point in which the line $4x_1 - 5x_2 + 3x_3 = 0$ cuts the line joining the two points $u_1 - u_2 + u_3 = 0$, $2u_1 + u_2 + 2u_3 = 0$.

13. Translate into a projective theorem concerning conics: The angle inscribed in a fixed arc of a circle is constant.

14. Find the equation of the polar point of the line $(2, 3, -1)$ with respect to the points $5u_1 - u_2 + u_3 = 0$, $u_1 + 2u_2 - u_3 = 0$.

15. Find the three polar lines of the point (a_1, a_2, a_3) with respect to pairs of sides of the triangle of reference. Show that these lines cut the sides of the triangle where they meet the polar of the point with respect to the triangle. Obtain thus a new definition of the polar line of a point as to a triangle. Dualize.

16. Find the homogeneous equation of (a) the point (7), (b) the line (11), §55, (c) the line whose parametric equations are

$$x_1 = 2at + a^2,\ x_2 = t + 2a,\ x_3 = a^2t + 1.$$

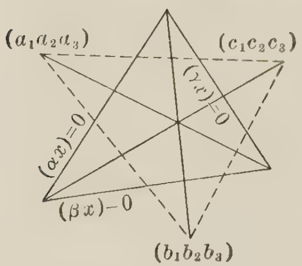
59. Perspective triangles.—Two plane n-points are *perspective from a point,* the center of perspection, if junctions of corresponding vertices are on the point. Dually two plane n-lines are *perspective from a line,* the axis of perspection, if intersections of corresponding sides are on the line. These definitions hold whether the figures are in the same plane or not. If the figures are in the same plane the center or axis is also in the plane.

The condition that two triangles in the same plane be perspective is found most conveniently when one is considered a 3-point and the other a 3-line. Let the equations of the three points and the three lines be respectively

$$(au) = 0,\ (bu) = 0,\ (cu) = 0, \tag{1}$$

$$(\alpha x) = 0,\ (\beta x) = 0,\ (\gamma x) = 0, \tag{2}$$

where the parentheses as usual indicate row products, thus $(au) \equiv a_1u_1 + a_2u_2 + a_3u_3$.



Then the equation of a line on the intersection of α and β is

$$(\alpha x) + \lambda(\beta x) = 0. \tag{3}$$

If this line be required to pass through (c_1, c_2, c_3) we have

$$(\alpha c) + \lambda(\beta c) = 0 \tag{4}$$

as an equation to determine λ. Whence the equation of the line is

$$(\alpha x)(\beta c) = (\beta x)(\alpha c). \tag{5}$$

From symmetry the equations of the lines joining the other pairs of corresponding vertices are

$$(\beta x)(\gamma a) = (\gamma x)(\beta a) \tag{6}$$

$$(\gamma x)(\alpha b) = (\alpha x)(\gamma b). \tag{7}$$

The condition that these equations have a common solution, *i. e.*, that the lines meet in a point is found by eliminating x.

Multiplying the left and right sides of the equations and removing the common factors we obtain

$$(\beta c)(\gamma a)(\alpha b) = (\alpha c)(\beta a)(\gamma b) \tag{8}$$

which is the condition that the triangles be perspective from a point.

Similarly the equations of the points of intersection of corresponding sides of the two triangles are

$$\begin{aligned}(au)(b\gamma) &= (bu)(a\gamma)\\ (bu)(c\alpha) &= (cu)(b\alpha)\\ (cu)(a\beta) &= (au)(c\beta).\end{aligned} \tag{9}$$

Eliminating u *the condition that these points be collinear, i. e., that the triangles be perspective from a line is*

$$(b\gamma)(c\alpha)(a\beta) = (a\gamma)(b\alpha)(c\beta), \tag{10}$$

the same condition as before. Combining this result with Ex. 5, §39, we have the important

Theorem of Desargues.—*Two triangles (in the plane or in space) which are perspective from a point are also perspective from a line, and conversely.*

The figure of two perspective triangles thus contains 10 points (the 6 vertices, the intersections of corresponding sides, and the center) and 10 lines (the sides of the triangles, junctions of corresponding vertices, and the axis) so situated that 3 of the points are on each line and 3 of the lines are on each point. Such a figure of n points and m lines so related that p of the points are on each of the lines and l of the lines are on each of the points is called a *configuration.* The configuration is sometimes represented symbolically by $\begin{vmatrix} n & l \\ p & m \end{vmatrix}$. The symbol for the configuration of two perspective triangles, called the *Desargues configuration* is thus $\begin{vmatrix} 10 & 3 \\ 3 & 10 \end{vmatrix}$.

60. Multiply perspective triangles.—Relation (8) of the previous section is not the most general condition that two triangles be perspective for it was derived on the assumption that the vertices were paired in a definite way. Now the vertices of two triangles can be paired in no fewer than six ways and the condition that they be perspective in any of these ways is a relation similar to (8) (transposed). The general condition for perspective triangles therefore is the product of the six factors obtained by pairing the vertices in all possible ways. The triangles will then be simply, doubly, . . . sextuply (fully) perspective according as 1, 2 . . . 6 of these factors = 0.

Denoting the vertices of the triangles by a, b, c and a', b', c' and the opposite sides by α', β', γ' and α, β, γ respectively we shall use the notation $\begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix}$ to designate perspective triangles where corresponding elements are those in the same column.[1] All possible arrangements are obtained most conveniently by permuting the letters in one row only.

If two triangles are perspective in the orders $\begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix}$ and $\begin{pmatrix} a & b & c \\ b' & c' & a' \end{pmatrix}$, *they are also perspective in the order* $\begin{pmatrix} a & b & c \\ c' & a' & b' \end{pmatrix}$. For translated into the language of conditions this is another way of saying

If

$$(\beta c)(\gamma a)(\alpha b) = (\alpha c)(\beta a)(\gamma b)$$

and

$$(\gamma c)(\alpha a)(\beta b) = (\beta c)(\gamma a)(\alpha b),$$

then

$$(\alpha c)(\beta a)(\gamma b) = (\gamma c)(\alpha a)(\beta b)$$

[1] Any pairing of vertices carries with it of course a definite pairing of the sides and dually. Thus in the triangles $\begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix}$ bc, ca, ab correspond respectively to $b'c'$, $c'a'$, $a'b'$.

which is obviously true. Note that the vertices of one triangle have been permuted cyclically, otherwise to be doubly perspective does not imply being triply perspective.

61. Heretofore we have supposed that the vertices of perspective triangles were two distinct sets of points. If this restriction is removed so that the same point may belong to different triangles, we have the theorem

If two triangles are perspective the six vertices can be grouped in four ways to form perspective 3-points.

For manifestly if the points form the perspective 3-points $\begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix}$ they can also be arranged according to the schemes

$$\begin{pmatrix} a' & b & c \\ a & b' & c' \end{pmatrix}, \begin{pmatrix} a & b' & c \\ a' & b & c' \end{pmatrix}, \begin{pmatrix} a & b & c' \\ a' & b' & c \end{pmatrix}.$$

While these four pairs of triangles have a common center of perspection, the four axes of perspection are distinct.

Again since the Desargues configuration is symmetrical with respect to all its points and lines, any point may be taken as a center of perspection. The six points lying on lines through this center may then be arranged as vertices of two perspective triangles whose axis contains the remaining three points. The configuration can thus be regarded in ten ways as made up of two perspective triangles. The six vertices in each of the ten arrangements can of course be broken up into four pairs of triangles with the same center of perspection.

EXERCISES

1. Write the general condition (product of six factors) that two triangles be perspective. Find this condition when one of the triangles is the reference triangle.

2. Show that if two triangles are perspective in five ways they are fully perspective. Find conditions that two triangles be quadruply

perspective which also make the triangles fully perspective. Reconcile this with the first part of the exercise.

3. Find the conditions that the simple polygons $(au)(bu)(cu)(du) = 0$ and $(\alpha x)(\beta x)(\gamma x)(\delta x) = 0$ be perspective from a point. Will they then be perspective from a line?

4. The maximum number of ways six points can be grouped into perspective triangles is ten.

5. How many constants are possessed by two simply perspective triangles? Two fully perspective triangles?

6. Construct a Desargues configuration. Select pairs of triangles perspective from each point and such that the axes of perspection will be the lines of the figure. Show that the configuration can be regarded as made up of a complete 4-point and a complete 4-line. Separate the points and lines of the configuration into two simple pentagons such that the vertices of each are on the sides of the other. (This can be done in six ways.)

7. Show that (*a*) when the axis of perspection is neglected two triangles perspective from a point solve the puzzle of planting 19 trees in 9 rows, 5 in a row; (*b*) when the axis is taken into account 25 trees can be planted in 10 rows, 6 in a row. The lines (rows) in these two figures should meet in 36 and 45 points respectively,—account for the others.

8. Construct the figure of the four pairs of triangles $\begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix}$, $\begin{pmatrix} a' & b & c \\ a & b' & c' \end{pmatrix}$, $\begin{pmatrix} a & b' & c \\ a' & b & c' \end{pmatrix}$, $\begin{pmatrix} a & b & c' \\ a' & b' & c \end{pmatrix}$. Show that the sides meet in six points which are the vertices of a complete quadrilateral (whose sides are the axes of perspection). Show that the diagonal triangle of this quadrilateral is perspective with any pair of the four triangles from their common center of perspection.

9. What are the symbols for the configurations of a complete 4-point, complete 5-point, complete n-point and their duals?

10. Show that the reference triangle and $(1, 1, 1)$, $(1, \omega, \omega^2)$, $(1, \omega^2, \omega)$ are fully perspective. Show that corresponding vertices lie in pairs on 9 lines which meet at 6 points (centers of perspection) and that the vertices, centers of perspection and the 9 lines therefore form the configuration $\begin{vmatrix} 12 & 3 \\ 4 & 9 \end{vmatrix}$. Find (*a*) the intersections of the sides of the triangles (*b*) the six centers of perspection. Show that these centers can be grouped to form two fully perspective triangles whose centers of perspection are the vertices of the original triangles. Prove

that any two of the four triangles are fully perspective and that the centers of perspection are the vertices of the other two. Show that the intersections of the sides of the first two triangles lie by threes on the twelve sides of the four triangles and that therefore the points and lines form the configuration $\begin{vmatrix} 9 & 4 \\ 3 & 12 \end{vmatrix}$. These two configurations are dual.

11. Define polar point and line with respect to a triangle (§28, Ex. 7) in the language of perspective triangles.

12. Consider the 4-point $(1, \pm 1, \pm 1)$. Show that the coördinate axes cut the sides of the 4-point (exclusive of the intersections at the diagonal points) in six points which are the intersections of the four lines $(1, \pm 1, \pm 1)$. Thus the six vertices of the 4-line are the sides of the 4-point. The polar of each point of the 4-point with respect to the triangle of the other three is a line of the 4-line. The 4-point and the 4-line have a common diagonal triangle. The figure is a special Desargues configuration and is self-dual. Draw the figure (*a*) beginning with the 4-point, (*b*) beginning with the 4-line.

13. When each point of a 4-point is omitted in turn, four triangles are obtained each perspective with the diagonal triangle, the center of perspection in each case being the isolated point. The axes of perspection are the sides of the 4-line associated with the 4-point (Ex. 12). Dualize.

14. Draw the figure in Ex. 12 when the 4-point is (*a*) a square, (*b*) a triangle and its centroid.

62. Transformation of coördinates in the plane.—It should be observed that the equation $f(x_1, x_2, x_3) = 0$ of any curve is expressed by means of the equations $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ of the sides of the reference triangle. Suppose now for example that a conic can be thrown into the form

$$(x_1 - x_2 + 2x_3)(2x_1 + x_2 - 3x_3) - (x_1 + x_2 + x_3)^2 = 0.$$

Then evidently we may replace the parentheses by x_1', x_2', x_3' respectively whereupon the conic takes the form

$$x_1'x_2' - x_3'^2 = 0$$

which is the equation of the curve referred to the triangle whose sides are

$$
\begin{aligned}
x_1' &= x_1 - x_2 + 2x_3 = 0\\
x_2' &= 2x_1 + x_2 - 3x_3 = 0\\
x_3' &= x_1 + x_2 + x_3 = 0.
\end{aligned}
$$

This suggests a method of passing from one projective coördinate system in the plane to another. Let (x_1, x_2, x_3) be the coördinates of a point in the original system while (x_1', x_2', x_3') are the coördinates of the same point in the new system. Then if the sides of the new triangle of reference expressed in terms of the original coördinates are

$$a_ix_1 + b_ix_2 + c_ix_3 = 0, \qquad i = 1, 2, 3 \tag{1}$$

the relation between the new and old coördinates is defined by the equations

$$
T^{-1}\colon\ \begin{aligned}
x_1' &= a_1x_1 + b_1x_2 + c_1x_3\\
x_2' &= a_2x_1 + b_2x_2 + c_2x_3\\
x_3' &= a_3x_1 + b_3x_2 + c_3x_3.
\end{aligned} \tag{2}
$$

To express the old coördinates in terms of the new we have but to solve the equations just written for x_1, x_2, x_3[1] thus

$$
T\colon\ \begin{aligned}
Ax_1 &= A_1x_1' + A_2x_2' + A_3x_3'\\
Ax_2 &= B_1x_1' + B_2x_2' + B_3x_3'\\
Ax_3 &= C_1x_1' + C_2x_2' + C_3x_3'.
\end{aligned} \tag{3}
$$

Where A_1, *e. g.*, is the cofactor of a_1 in the determinant

$$A \equiv \begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3 \end{vmatrix}$$

T and T^{-1} are the desired formulas for the transformation of coördinates. T^{-1} carries the old coördinates x of a

[1] This is always possible since the determinant $A \neq 0$, the new triangle of reference being by hypothesis a proper triangle.

point into the new coördinates x'. If however we wish to transform the equation $f(x_1, x_2, x_3) = 0$ of a *curve* referred to the old triangle into its equation with respect to the new triangle we must apply formulas T. To reverse either operation we merely interchange the two sets of formulas, T and T^{-1} being inverse to each other.[1]

Dually if the coördinates of a line referred to the 3-point $u_1 = 0$, $u_2 = 0$, $u_3 = 0$ are (u_1, u_2, u_3) and the coördinates of the same line referred to the 3-point

$$a_iu_1 + b_iu_2 + c_iu_3 = 0, \qquad i = 1, 2, 3 \tag{4}$$

are (u_1', u_2', u_3'), then the equations which transform the old coördinates u of a line into the new coördinates u' are

$$S^{-1}: \begin{aligned} u_1' &= a_1u_1 + b_1u_2 + c_1u_3 \\ u_2' &= a_2u_1 + b_2u_2 + c_2u_3 \\ u_3' &= a_3u_1 + b_3u_2 + c_3u_3. \end{aligned} \tag{5}$$

While to pass from the equation of a curve $f(u_1, u_2, u_3) = 0$, referred to the old triangle, to the equation referred to the new triangle we employ the formulas

$$S: \begin{aligned} Au_1 &= A_1u_1' + A_2u_2' + A_3u_3' \\ Au_2 &= B_1u_1' + B_2u_2' + B_3u_3' \\ Au_3 &= C_1u_1' + C_2u_2' + C_3u_3,' \end{aligned} \tag{6}$$

where the coefficients have the same significance as before. S and S^{-1} are likewise transformations inverse to each other. Thus S changes the new *coördinates* u' back into

[1] Strictly since the choice of the reference lines does not suffice to determine the coördinate system completely, the equations T^{-1} should be written

$$x_i' = k_i(a_ix_1 + b_ix_2 + c_ix_3), \qquad i = 1, 2, 3$$

where the k_i are to be determined by the choice of the unit point in the new system. This will modify the form of the transformed equations somewhat but it will not materially effect the study of the projective properties of curves. A similar statement of course applies to the dual case below.

the old coördinates u while S^{-1} transforms the equation of a *curve* referred to the new triangle into its equation referred to the original triangle.

Effect of a transformation of coördinates.—Since all the formulas for the transformation of coördinates are linear in the variables involved, neither the order nor the class of a curve is altered. In particular a point is transformed into a point and a line into a line. It follows that a range of points goes into a range and a pencil of lines into a pencil. We shall now prove that

The double ratio of four collinear points is independent of the coördinate system.

If $(\alpha_1, \alpha_2, \alpha_3)$ and $(\beta_1, \beta_2, \beta_3)$ are the coördinates of any two points in the original system then the coördinates of a variable point x collinear with them are (§55) $x_i = \alpha_i + \lambda\beta_i$, $i = 1, 2, 3$. And the double ratio of four points x is equal to the double ratio of their four parameters. Now under the transformations T^{-1} the coördinates x are changed into coördinates x' in the new system thus

$$\begin{aligned} x_i' &= \alpha_i' + \lambda\beta_i' \\ &\equiv a_i\alpha_1 + b_i\alpha_2 + c_i\alpha_3 + \lambda(a_i\beta_1 + b_i\beta_2 + c_i\beta_3). \end{aligned}$$

The double ratio of four points x' is likewise equal to the double ratio of their four parameters. But the parameter of each point x is identical with that of the corresponding x', which proves the theorem. In short

Projective properties are unaltered by any change in the coördinate system.[1]

The definition of projective coördinates (§51) may be regarded as a transformation of the Cartesian coördinates

[1] Which justifies the name projective coördinates. It is otherwise however with metric properties. For example by the very simple transformation $x' = 2x$, $y' = y$, the circle $x^2 + y^2 = a$ is changed into an ellipse. The only transformations that preserve both the size and shape of a figure are (*a*) the translation, (*b*) the rotation of axes for the axes must remain perpendicular and the scale of measurement must be unaltered.

of a point into projective coördinates so that the transformations considered in the preceding section include the transformation of Cartesian coördinates as a special case.

Again if we wish to give a metrical setting to a curve whose equation in projective coördinates is $f(x_1, x_2, x_3) = 0$ we may interpret the variables as homogeneous Cartesian coördinates. Then writing $x_1 = x$, $x_2 = y$, $x_3 = 1$ we obtain the ordinary rectangular form. This is not a legitimate transformation of coördinates. The effect is however the same as if two sides of the triangle of reference had been revolved about a vertex until they became perpendicular while the third side x_3 was sent off to infinity. All points on x_3,—whether they be ordinary points, points of inflexion, double points, etc.—are of course carried off to infinity. Tangents to f with contacts lying on $x_3 = 0$ become asymptotes, the point pair $u_1^2 + u_2^2 = 0$ (in projective line coördinates) go into the circular points while intersections of tangents from this pair of points to the curve become foci of the curve. In the same way any side of the triangle of reference may be isolated for the line at infinity and the other two taken as rectangular axes.[1]

On the other hand we may conveniently study the behavior at infinity of a curve whose rectangular equation is $F(x, y) = 0$ by first making F homogeneous in the Cartesian coördinates x, y, z (§28) and then supposing the variables to represent projective coördinates by the substitutions $x = x_1$, $y = x_2$, $z = x_3$. While not a transformation of coördinates in the sense of this article the effect is as if $\mathfrak{L}$ had been brought into the finite region of the plane. The

[1] In the ordinary transformation of coördinates we deal with two triangles of reference which have distinct projective relations to the curve. Whereas the present process is really equivalent to a projection of the curve *together with the reference triangle*,—it being possible in a single projection to send any line into $\mathfrak{L}$ and any angle into any other,—which explains the persistence of the pro'ective relation of the curve to its triangle of reference. See below, §161, Ex. 5.

two curves indeed have the same projective relations to their respective triangles. Metrical properties of F become simply properties with special relations to the line $x_3 = 0$. Asymptotes correspond to tangents with contacts on x_3, parallel tangents go into tangents meeting on x_3, etc.

EXERCISES

1. The cofactor of A_1 in the determinant

$$A \equiv \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

is a_1A.

2. If the triangle of reference in the dual transformation (§62) is the same as that used for point coördinates the equations of transformation are

$$S^{-1}\colon u_i' = A_iu_1 + B_iu_2 + C_iu_3, \qquad i = 1, 2, 3,$$
$$S\colon u_1 = (au'), \qquad u_2 = (bu'), \qquad u_3 = (cu').$$

3. Find the equations for transformation of coördinates (point and line) when the equations of the sides of the old triangle expressed in terms of the new coördinates are

$$a_ix_1' + b_ix_2' + c_ix_3' = 0, \qquad i = 1, 2, 3.$$

4. The formulas for the transformation of coördinates contain eight essential constants.

5. Show that the coördinates of four points (no three on a line) in the one system can be assigned arbitrary names in the new system (provided only that the new names do not imply collinearity of three of the points) which completely determines the transformation.

6. The transformation of coördinates is completely determined when the new triangle of reference and the new unit point are chosen.

7. Determine the transformation that changes the coördinates of the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ respectively into $(2, 1, -1)$, $(-1, 2, 1)$, $(1, -1, 2)$ and leaves the unit point unaltered.

8. Determine the transformation that changes the names of the points $(0, 0, 1)$, $(1, i, 0)$, $(1, -i, 0)$, $(1, 1, 1)$ respectively to $(0, 0, 1)$, $(0, 1, 0)$, $(1, 0, 0)$, $(1 + i, 1 - i, 1)$, $i = \sqrt{-1}$.

9. Show that the equations for translation and rotation of axes in rectangular coördinates are special cases of the transformations in this section.

10. If the rectangular axes are rotated through an angle θ the equations of the new axes are $y + x \cot \theta = 0$, $y - x \tan \theta = 0$, or $x \cos \theta + y \sin \theta = 0$, $-x \sin \theta + y \cos \theta = 0$. The equations of transformation are thus

$$\begin{aligned} x' &= x \cos \theta + y \sin \theta \\ y' &= -x \sin \theta + y \cos \theta. \end{aligned}$$

If we set x' and y' equal to the first pair of equations, what would be the effect on the coördinates? What would be the effect of changing the sign of either or both equations?

CHAPTER VI

THE CONIC

63. Projective generation of conics.—We have seen (§40) that the intersections of corresponding lines of two perspective pencils lie on a line, the axis of perspectivity. Consider now two projective pencils. Since each pencil is a singly infinite system of lines the intersections of corresponding lines will constitute a one-parameter family of points, *i. e.*, a point curve. The question is what sort of curve is thus defined? We shall prove the following

THEOREM. *The locus of intersections of corresponding lines of two projective pencils is a conic passing through the centers of the pencils.*

Suppose the equations of the two pencils are

$$(ax) + \lambda(bx) = 0 \tag{1}$$

and

$$(cx) + \lambda(dx) = 0 \tag{2}$$

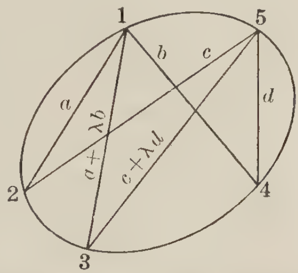
where

$$(ax) \equiv a_1x_1 + a_2x_2 + a_3x_3, \text{ etc.}$$

These pencils will be projective if we correlate lines having the same parameter. For the pencils are in (1, 1) correspondence since every value of λ gives one and only one line of each pencil. Moreover the double ratio of four lines $(ax) + \lambda_i(bx) = 0$ is equal to the double ratio of the four $(cx) + \lambda_i(dx) = 0$, $i = 1, 2, 3, 4$, each being equal to $(\lambda_1\lambda_3|\lambda_2\lambda_4)$.

Eliminating λ from (1) and (2) the locus of intersections of corresponding lines is found to be

$$(ax)(dx) - (cx)(bx) = 0. \tag{3}$$



Since each parenthesis is linear in x_1, x_2, x_3 this equation is of the second degree and therefore represents a conic. To see that the conic is on the center of pencil (1) it is only necessary to observe that this center lies on both the lines a and b and hence that its coördinates satisfy their equations. But any numbers that cause (ax) and (bx) to vanish will also satisfy (3), *i. e.*, the center is on the conic. Likewise the other center is a point of the conic.

If the two pencils are perspective their common line, *i. e.*, the line joining the centers, is self-corresponding. Taking this line as $\lambda = 0$ we have $(ax) \equiv (cx)$ and the equation of the locus reduces to $(ax)\{(dx) - (bx)\} = 0$. Thus when the pencils are perspective the locus (3) degenerates into the line of centers and another line, the axis of perspection.

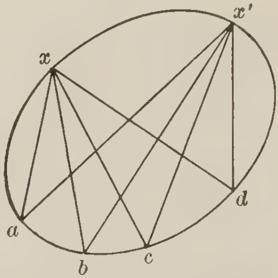
That (3) is a general conic appears from its equation which contains five essential constants. It is also evident geometrically that the general conic may be described by this method. For the centers of the two pencils (say points 1 and 5) can be selected at random. Then any other three points 2, 3, 4 can be taken as intersections of corresponding lines from 1 and 5. The locus generated by the projective pencils so determined will pass through the five arbitrary points 1, 2, 3, 4, 5; it is therefore a general conic.

This theorem enables us *to construct point by point the conic determined by five given points.* For three pairs of corresponding lines like 12, 13, 14 and 52, 53, 54 above estab-

8

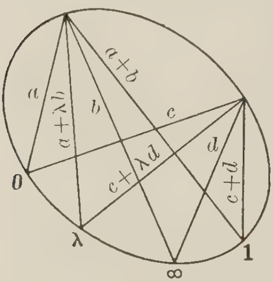
lish a projectivity between the two pencils with centers 1 and 5. We may then construct pairs of corresponding lines at pleasure (§50, 5°, dual) which will meet at points of the conic.

64. Cor. *If four fixed points on a conic are joined to a variable fifth, the double ratio of the pencil so formed is constant.* For if x and x' denote the variable point in two positions while a, b, c, d, represent the four fixed points, then the lines x-$abcd$ and x'-$abcd$ are corresponding lines of two projective pencils which generate the conic. Hence the double ratio of the two pencils is the same.



Q.E.D.

We have thus a double ratio which is characteristic of four points on a conic and which may be called the double ratio of the points. This affords a *method of naming the points on a conic by a single coördinate.* We saw that corresponding lines in the pencils which generate a conic (§63) are those with the same value of λ. The value of λ common to the two lines may be considered as the *coördinate* or parameter of their point of intersection which is a point of the conic.



Now taking as reference points on the conic the intersections of the base lines of the two pencils and the unit lines, *i. e.*, the points 0, ∞, 1 we have as the double ratio of these points and any fourth point λ, $(0\ \infty \mid \lambda 1) = \lambda$.

We have thus spread a parameter λ along the conic and hence established a (1, 1) correspondence between the points of a conic and the points of a line.[1]

From the definitions it is evident that the double ratio of four points on a conic, as in the case of the line, is equal to the double ratio of their parameters.

EXERCISES

1. Dualize §63.

2. If S and S' are centers of two projective pencils which generate a conic show that SS' considered as a line of the pencil on S has as correspondent the tangent to the conic at S' and vice versa.

3. From the result of Ex. 2 show how to construct a tangent at any point of a conic.

4. Given five points show how to construct at any of the points the tangent to the conic through the five.

5. Dualize Exs. 2, 3, 4.

6. Construct a conic when one of the pencils is a parallel pencil; when both are.

7. Show how a given conic can be generated by two projective pencils; two projective ranges.

8. Construct a line conic when one of the generating ranges is the line at infinity.

9. Dualize the corollary of §64.

10. The locus of a line which cuts four fixed lines in a constant double ratio is a line conic tangent to the four lines. Dualize.

11. Find the equations of the conics generated by

$$(a)\ \begin{array}{l} y = tx \\ y = 1/t, \end{array} \qquad (b)\ \begin{array}{l} tx = k \\ y = t, \end{array} \qquad (c)\ \begin{array}{r} x + iy + tz = 0 \\ t(x - iy) + z = 0. \end{array}$$

12. Prove the first theorem of §63 synthetically. (Show that the locus cannot cut an arbitrary line in more than two points.)

[1] For the points of a line can be named by a single coördinate λ. Corresponding points of line and conic are those having the same value of λ. Or the correspondence may be established geometrically by taking the center of a line pencil on the conic. Lines of the pencil will then cut an arbitrary line and the conic in pairs of corresponding points. The same value of λ may then be assigned to corresponding points so determined.

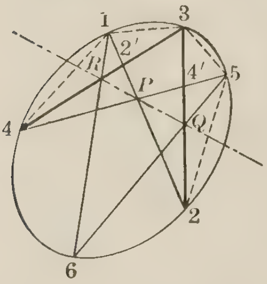
65. Pascal's theorem.—Six points on a conic joined in order constitute a simple hexagon inscribed in the conic. If the points are 1, 2, 3, 4, 5, 6 we may designate 12 45, 23 56, 34 61 as opposite sides. Let these meet respectively in P, Q, R indicated by the scheme

$$\begin{array}{ccc} 12 & 45 & P \\ 23 & 56 & Q \\ 34 & 61 & R. \end{array}$$

We shall now prove the famous theorem first announced by Pascal.

The opposite sides of a hexagon inscribed in a conic meet in three points of a line.[1]

Synthetic Proof.—Suppose the conic is generated by pencils with centers at 1 and 5. Then 12, 52 etc. are pairs of corresponding lines. Now corresponding lines of these pencils will cut any two lines, say 34 and 23 respectively, in corresponding points of projective ranges. We have thus (see figure) $2'4\ R\,3 \overline{\wedge} 2\,4'\,Q\,3$. But the two ranges have a self-corresponding point 3; they are



[1] This theorem marks the climax of the classical theory of projective geometry. Its importance in the synthetic treatment of conics can hardly be exaggerated. But it has enjoyed a popularity commensurate with if not exceeding its importance. Discovered by its precocious author at the age of 16, studied by many of the eminent fathers of projective geometry, this theorem caught the imagination of mathematicians to an astonishing degree. As the remarkable properties of the complete 6-point were unfolded, men called it in their enthusiasm the *mystic hexagram* (*hexagramma mysticum*). This is perhaps not surprising in view of the possibility of drawing with the aid of the theorem elements of a conic such as a tangent at a point, tangents from a point, asymptotes, center, etc. when the conic itself is represented only by a skeleton of five points! At one time however it became almost a menace to mathematical progress, investigators turning away from their search for new truth to devote themselves to finding new proofs of Pascal's theorem.

therefore perspective[1] and $22'$, $44'$ and RQ, *i. e.*, 12, 45 and RQ meet in a point P,

Q.E.D.

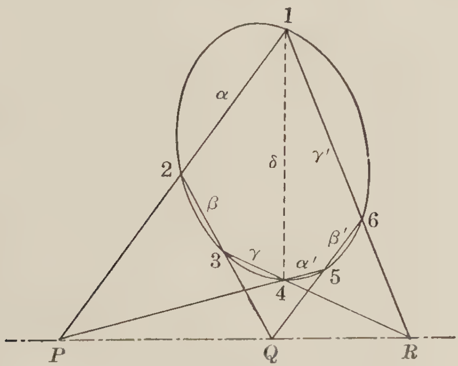
Analytic Proof.—Lemma. If $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\delta = 0$ are the equations of the sides of a simple quadrilateral then

$$\alpha\gamma + k\beta\delta = 0$$

is the equation of a conic on the vertices of the quadrilateral. For the equation is of the second degree and it is satisfied when $\alpha = 0$, $\beta = 0$, *i. e.*, by the intersection of α and β. And so for the other vertices.

Now denote the sides of the hexagon as in the figure and let δ be the line 14. Then the equation of the conic since it circumscribes the two quadrilaterals may be written in the identical forms

$$\alpha\gamma + k\beta\delta = 0$$
$$\alpha'\gamma' + k'\beta'\delta = 0.$$



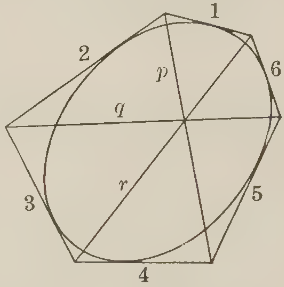
Therefore the two sides of the identity

$$\alpha\gamma - \alpha'\gamma' \equiv \delta(k'\beta' - k\beta),$$

set equal to zero, represent the same locus which is obviously

[1] §50, 4°, Cor.

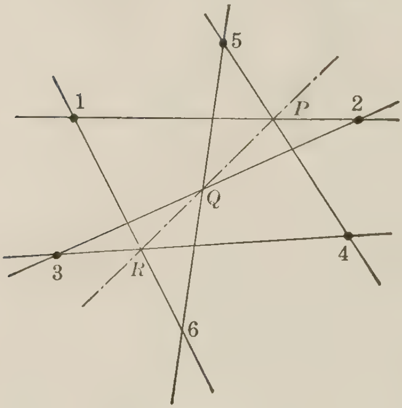
a pair of lines. Now the locus on the left passes through the vertices of the quadrilateral $\alpha\alpha'\gamma\gamma'$ (lemma) and hence consists of the diagonals δ and PR of the quadrilateral. It follows that the line $k'\beta' - k\beta = 0$ joins the points P, R. And it is evident from the equation that it also contains Q; in other words P, Q, R are collinear. PQR is called the *Pascal line* of the hexagon.



The dual of Pascal's theorem is known as

Brianchon's theorem.[1]—*The opposite vertices of a hexagon circumscribing a conic lie on lines through a point.* The point is the *Brianchon point* of the hexagon.

66. Pascal's theorem supplies a convenient method of constructing linearly any number of points of a conic on five given points. Let the points be 1 2 3 4 5. Draw 12 and 45 meeting at P. Then through 1 draw any line 16 on which a sixth point 6 is to be located. This line cuts 34 in R. Next draw Pascal's line PR meeting 23 in Q. Finally $5Q$ will intersect 16 in the required point 6.



It should be observed that we have solved a more specific problem than the mere construction of random points

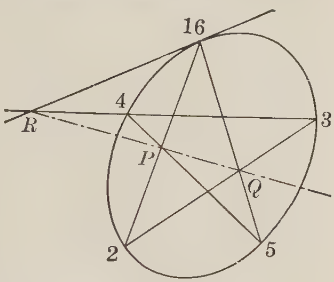
[1] It is a remarkable fact that while Pascal's theorem was published in 1640 Brianchon's did not appear till 1806. Needless to say the principle of duality was unknown at the earlier date.

on the conic; we have really found the second point in which a line on one of the base points cuts the conic. We can thus study the figure of the curve in any region for the lines 16 may be drawn in any desired direction. It goes without saying that there is no essential modification in the construction if 1 and 6 are replaced by any two of the consecutive points.

Special Cases of Pascal's Theorem

67. Pascal's theorem is still valid when one or more pairs of consecutive vertices coincide. The line joining such a pair of coincident vertices is then a tangent to the conic. We obtain thus three corollaries.

Cor. 1. $1 \equiv 6$. *If a simple pentagon* 1 2 3 4 5 *be inscribed in a conic the pairs of lines* 12 45, 23 51(56), 34 *and the tangent* (16) *at* 1 *meet in three points of a line.* Similar statements of course apply to the other vertices.



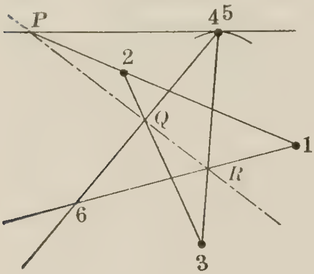
Problem (*a*). This theorem enables us, *given five points, to draw at any one of them the tangent to the conic determined by the points.* For to draw the tangent at 1 is to construct the line (16) joining the coincident vertices 1 and 6. Thus 12 45 meet in P, 23 51(56) meet in Q determining Pascal's line. 34 cuts this line in R and $R1$ is the required tangent (16).

The corollary also furnishes a solution to another important

Problem (*b*). *Given four points of a conic and the tangent at any one of them, to construct the conic.*

Let 1 2 3 4 be the points and 45 the tangent at $4 \equiv 5$.

Then 12 cuts 45 in P. On 1 draw any line 16 cutting 34 in R. Draw Pascal's line PR meeting 23 in Q. Then $5Q$ intersects 16 in the required point 6.



Here again we can control the construction in a measure by drawing 16 in any desired direction. Or the problem may be otherwise stated: Given four points on a conic and a tangent at any one of them. To find the second point of intersection with the conic of any line on any one of the base points.

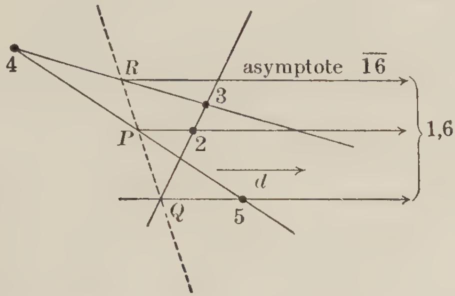
Problems (a) and (b) have a variety of metrical aspects when one or more of the elements involved are at infinity. Thus if one of the points is at infinity the tangent at the point will be an asymptote and lines drawn to the point will be parallel to the asymptote. In other words *to be given the direction of an asymptote is equivalent to being given a point at infinity on the conic.*

To be given an asymptote is to be given a point at infinity and the tangent at the point.

Likewise to stipulate that the conic is a parabola is equivalent to fixing a tangent (£). While to be given a parabola and the direction of the axis is to be given a tangent (£) and the contact of the tangent.

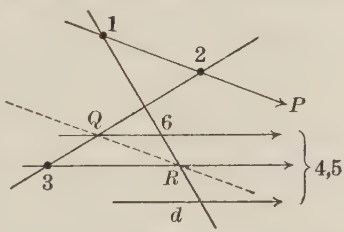
As an example of problem (a) metrically stated, suppose we are given four points on a hyperbola and the direction of an asymptote, to draw the asymptote. Take as the four points 2 3 4 5 and let the direction of the asymptote be indicated by the arrow d. Then d cuts the line at infinity in the coincident points 1, 6. Draw 12 (*i. e.*, a line through 2 parallel to d) meeting 45 in P. Likewise 23 and 56 (a line on 5 parallel to d) intersect in Q. P and Q determine

Pascal's line which cuts 34 in R. Finally $R1$ (a line on R parallel to d) is the required asymptote 16.



Observe that this construction is the same as that for the general case of problem (a) when the contact $1(\equiv 6)$ of the tangent line 16 has receded to infinity in the direction of the arrow d.

We shall now solve a typical metrical case of problem (b). If the given tangent is the line at infinity the conic becomes a parabola whose contact with $\mathcal{L}$ is in the direction of the axis. We may then state the case as follows. *Given three points of a parabola (in the finite region) and the direction of the axis, to construct the curve.*



Let 1 2 3 be the given points and let 4, 5 coincide at infinity in the direction of the arrow d.[1] Draw 12, the infinite point of which will be point P. On 1 draw any line 16 meeting 34 (a line on 3 parallel to d) at R. Now draw Pascal's line (on R parallel to 12) cutting 23 in Q.

[1] *I.e.* line 45 is the line at infinity. In the previous example 1 and 6 coincide on the line at infinity but the line 16 is a finite line, the asymptote.

Then $Q5$ (a line on Q parallel to d) intersects 16 in a point 6 of the parabola. Similarly other points of the curve can be constructed at will.

This construction corresponds to the general case of problem (b) when the tangent line 45 moves off to infinity so that the contact 4, 5 is in the direction of the arrow d.

Other metrical problems that can be solved by repeated applications of (a) and (b) are

(1) Given four points on a hyperbola and the direction of an asymptote, to draw the tangent at each of the points.

(2) Given three points on a hyperbola and the directions of both asymptotes, to construct the asymptotes and the tangents at the three points.

(3) Given three points of a hyperbola, the tangent at one of them and the direction of one asymptote, to find the direction of the other asymptote.

(4) To construct a hyperbola when given

- (a) three points and an asymptote
- (b) three points, a tangent at one and the direction of an asymptote
- (c) two points, a tangent at one and the directions of both asymptotes
- (d) two points, one asymptote and the direction of the other asymptote.

The student should solve all these problems. Any convenient notation may be used for the special hexagon but it is essential that coincident vertices be consecutive, as $1 \equiv 2$ or $3 \equiv 4$ etc.

If side $1 \equiv 6$ in Brianchon's hexagon we have the

Dual of Corollary I. *If a simple pentagon be circumscribed to a conic the junctions of the three pairs of points* 12 45, 23 51, 34 *and the contact of tangent* 1 *lie on three lines through a point.*[1]

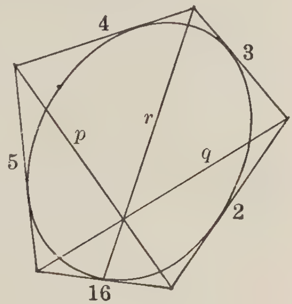
We have thus solutions of two problems (a') and (b'), duals of (a) and (b)

[1] See fig., next page.

(a') Given five lines, to find the contact of any one (*i. e.*, each) of them with the conic determined by the lines.

(b') Given four tangents of a conic and the contact of any one of them, to construct the conic by lines.

We append some metrical versions of these problems whose solutions are left to the student:



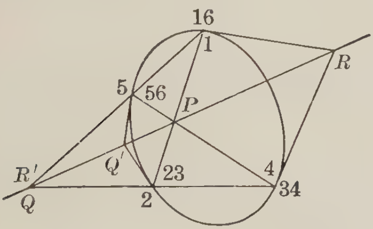
(1) Given four tangents to a parabola, to find the contacts of the tangents and the direction of the axis.

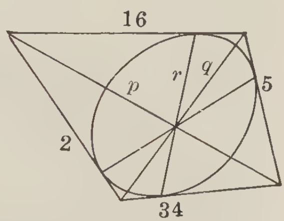
(2) Given three tangents to a parabola and the direction of the axis (or the contact of one of the tangents), to construct the parabola by lines.

(3) Given three tangents and an asymptote, to construct a hyperbola by lines.

68. Cor. 2. $1 \equiv 6$, $3 \equiv 4$. The hexagon is now a quadrangle and one pair of opposite sides (16, 34) are tangents to the conic. But Pascal's line is the same if we suppose vertices $2 \equiv 3$ and $5 \equiv 6$. The theorem then becomes

The opposite sides of a quadrangle inscribed in a conic together with the pairs of tangents at opposite vertices meet in four collinear points.





Dually the opposite vertices of a quadrilateral circumscribed to a conic and the contacts of opposite sides lie in pairs on concurrent lines.

These theorems furnish the solutions to two problems (*a*) and (*b*) and their duals (*a*′) and (*b*′). The student should make the appropriate constructions.

(*a*) Given four points on a conic and a tangent at one of them, to draw the tangent at each of the other points.

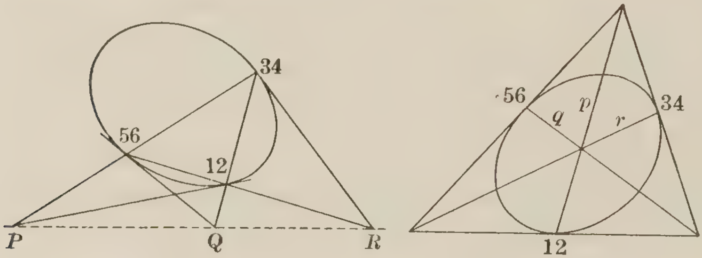
(*b*) Given three points on a conic and the tangents at two of them, to construct the conic by points.

(*a*′) Given four tangents of a conic and the contact of one of them, to find the contacts of the others.

(*b*′) To construct a conic by lines, having given three tangents and the contacts of two of them.

69. Cor. 3. If the six points coincide in pairs, $1 \equiv 2$, $3 \equiv 4$, $5 \equiv 6$, Pascal's theorem says

If a triangle is inscribed in a conic, the tangents at the vertices meet the opposite sides in three points of a line.



Dually the corresponding case of Brianchon's theorem is

If a triangle is circumscribed to a conic, the vertices and the contacts of the opposite sides lie in pairs on three concurrent lines.

Or the two may be combined into the single self-dual statement: *Three tangents to a conic and their points of contact form two perspective triangles.*

This corollary furnishes a solution to the dual problems (*a*) and (*a*′).

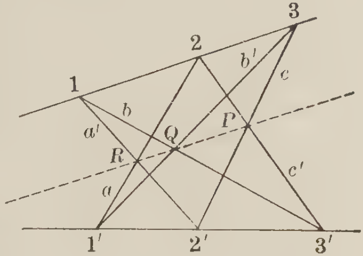
(a) Given three points on a conic and the tangents at two of them, to construct the tangent at the third.

(a') Given three tangents to a conic and the contacts of two, to construct the contact of the third.

70. Cor. 4. In the preceding cases we have stated the consequences of Pascal's theorem on the hypothesis that the hexagon is specialized. Suppose now that the conic itself degenerates. We then obtain an important special case known as the

Theorem of Pappus. *If* 1 2 3 *are three points on a line and* 1′ 2′ 3′ *are points of a second line then the pairs* 23′ 2′3, 31′ 3′1, 12′ 1′2 *intersect in three collinear points.*

For the pairs as written are opposite sides of the hexagon 23′12′31′ inscribed in the conic (line pair).



It is interesting to note that the theorem of Pappus is merely another form of the theorem concerning triply perspective triangles (§60). For beginning with 23′ and denoting the sides of the hexagon in order by $c'ba'cb'a$ the theorem of Pappus may be stated: If the triangles $\begin{pmatrix} a & b & c \\ b' & c' & a' \end{pmatrix}$ are perspective (from a line 1′2′3′) and also in the scheme $\begin{pmatrix} a & b & c \\ c' & a' & b' \end{pmatrix}$ (*i. e.*, from the line 123) then they are perspective in the order $\begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix}$ (from the line PQR).

EXERCISES

1. Prove Brianchon's theorem both synthetically and analytically.

2. The necessary and sufficient condition that the six points 1 2 3

4 5 6 lie on a conic is that the pairs of lines 12 45, 23 56, 34 61 meet in three collinear points.

3. If two triangles are perspective from a line PQR, the intersections of the sides of one with the *non-corresponding* sides of the other are six points of a conic. Denote the sides of the hexagon formed by joining the points in order by $a\ b'\ c\ a'\ b\ c'$. Then the perspective triangles are $\begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix}$.

4. Show that six points on a conic joined in all possible ways give rise to 60 simple hexagons and therefore 60 Pascal lines. Show that the 15 lines connecting the six points meet in 45 points like P, Q, R (exclusive of the original six) and that on each of the 45 points are four Pascal lines.

5. Dualize 2, 3, 4.

6. Dualize the construction of §66.

7. Dualize problems (a) and (b) §67.

8. By the aid of Pascal's and Brianchon's theorems and special cases solve the following problems:

(a) Given four points on a hyperbola and the direction of one asymptote to find the direction of the other asymptote.

(b) Given five tangents to a conic to draw the tangent parallel to each of them.

(c) Given four tangents to a parabola to draw from a point P on any of them the other tangent from P.

(d) Given four tangents to a parabola to construct the tangent parallel to a given line.

(e) To construct by lines the parabola determined by four given lines.

(f) To construct by points a hyperbola when given two points of the curve, their tangents and the direction of one asymptote; or one point of the curve, its tangent, one asymptote and the direction of the other; or one point on the curve and both asymptotes.

(g) To construct by points a parabola, having given two of its points, the tangent at one of them and the direction of the axis.

(h) To construct by lines a hyperbola, given one asymptote, two tangents to the curve and the contact of one of them; or both asymptotes and one tangent.

(i) To construct by lines a parabola, given two tangents, the

contact of one and the point at infinity on the curve; or two tangents and their contacts.

(*j*) Given two points on a hyperbola, their tangents and the direction of an asymptote, to construct the asymptote.

(*k*) Given one point of a hyperbola, its tangent, one asymptote and the direction of the other, to draw the second asymptote.

(*l*) Given both asymptotes of a hyperbola and one point of the curve, to draw the tangent at the point. Show from this construction that the segment of a tangent to a hyperbola cut off by the asymptotes is bisected by the point of contact.

(*m*) Given the direction of the axis of a parabola, two points of the curve and the tangent at one of them, to construct the tangent at the other.

(*n*) Given an asymptote of a hyperbola, two tangents and the contact of one, to find the contact of the other.

(*o*) Given both asymptotes of a hyperbola and one tangent, to find the contact of the tangent.

(*p*) Given two tangents of a parabola and their contacts, to determine the direction of the axis.

(*q*) Given the direction of the axis of a parabola, two tangents and the contact of one to locate the contact of the other.

Pole and Polar

71. Returning to Corollary 2 of Pascal's theorem, let the vertices of the (simple) quadrilateral 1 2 3 4 be cut from the conic by lines 13 and 24 on any point X (either figure).

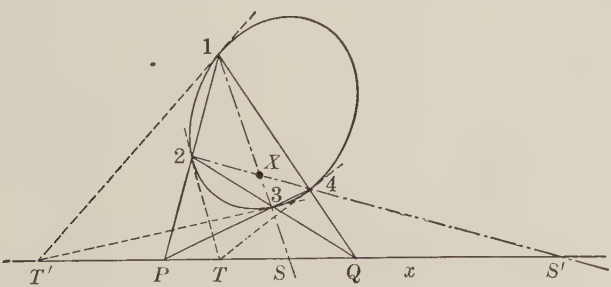


Fig. *a*.

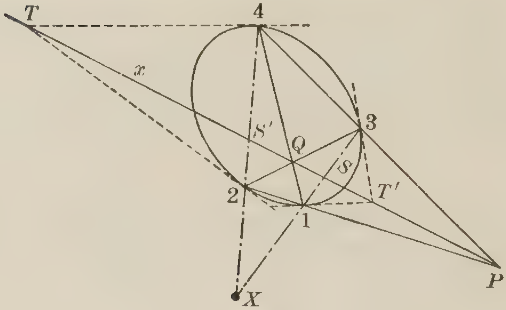


FIG. b.

Then we saw that the intersections (P, Q) of opposite sides, and the intersections (T, T') of tangents at opposite vertices are points of a line x. If the diagonal lines 13 and 24 cut this line in S and S' then 1, 3 and S, X as well as 2, 4 and S', X are harmonic pairs. Now hold fast one diagonal 13 and let the other revolve about X. An infinite system of quadrangles is thus generated with 1, 3 as a pair of opposite vertices. Accordingly for all quadrangles of the system the point T' is fixed and so is S (since $(1\ 3 \mid X\ S) = -1$), hence the line x is fixed. x is thus the locus of points P, Q, S' and T. Likewise if 24 is fixed, points T and S' are stationary and x is generated by the moving points P, Q, S and T'. When 13 (or 24) is tangent to the conic, T' (or T) is the point of contact.

x is called the *polar line* or *polar* of X with respect to the conic. We have the following equivalent definitions of the polar of a point X with respect to a conic according to the particular way we conceive it to be generated.

(a) The locus of points (S, S') harmonically separated from X by the extremities of a variable chord on X.

(b) The locus of the points of intersection (T, T') of the tangents at the extremities of a variable chord on X.

(c) The locus of intersections (P, Q) of pairs of opposite

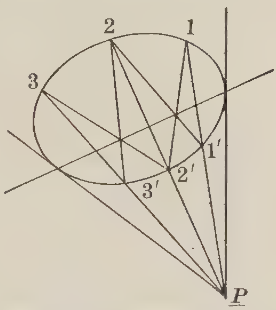
sides of an inscribed quadrilateral whose diagonal lines meet at X.

(*d*) The chord of contact of tangents[1] from X (Fig. *b*).

72. The construction of the polar line of a point as indicated in the preceding paragraph yields by a slight extension two other polar lines. For by considering in turn the three simple quadrilaterals determined by the four points 1 2 3 4 it is evident that PX is the polar of Q and QX is the polar of P. Or we may say

If the vertices of a complete 4-point lie on a conic each side of the diagonal triangle is the polar of the opposite vertex with respect to the conic.

The construction gives a convenient method of drawing the tangents to a conic from an exterior point since the polar line is then the chord of contact of tangents from the point. Therefore we need only connect the given point with the points in which the polar cuts the conic. Or we may state the solution in the form of a theorem, a special case of which has already been noticed (§46), the conic then being a pair of lines.



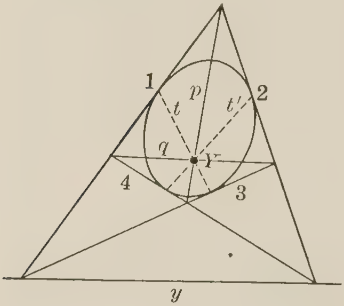
If 1, 1′; 2, 2′; 3, 3′ *etc. are pairs of points on a conic collinear with a fixed point* P, *then the pairs of cross lines* 12′, 1′2, *etc. meet in points of a line, the polar line of* P.

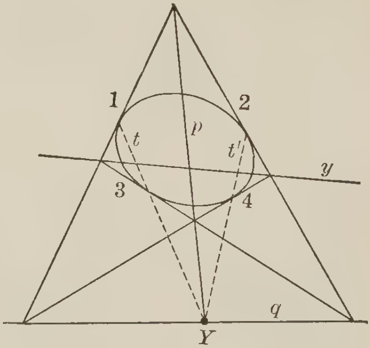
73. Polar point of a line.—We shall now consider the dual of §71. Given a line y, any two of its points determine two pairs of lines of a conic 1, 3 and 2, 4 which may be

[1] The contacts of tangents are real only when the point is *outside* the conic, which may be taken as a definition of "outside." When the point is inside the conic the contacts are conjugate imaginary but we shall see that they still lie on the polar line.

regarded as opposite sides of the simple circumscribed quadrilateral 1 2 3 4 (see figures).

Then the locus of





(*a*) lines harmonically separated from y by the tangents from a variable point of y

(*b*) the chord of contact (t or t') of the pair of tangents from a variable point of y

(*c*) junctions (p or q) of opposite vertices of a circumscribed quadrilateral whose diagonal points are on y

(*d*) is a point Y which is the intersection of tangents at the points where y cuts the conic.

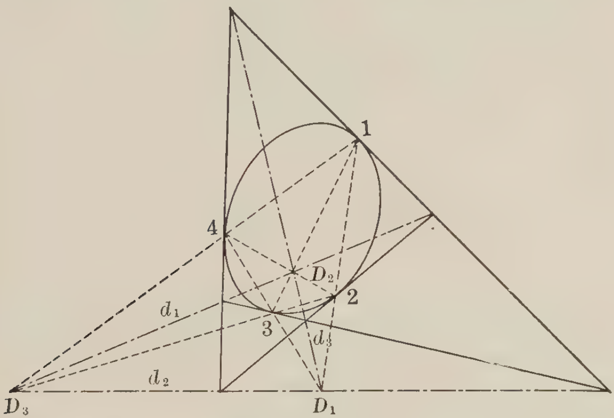
This point is called the *polar point* or *pole* of y with respect to the conic. *Pole and polar are thus dualistic.*

If now the line y be taken identical with x (§71) it appears that $Y \equiv X$. For the pole of x is the intersection X of the chords of contact of tangents from T and T' (from (*b*) above). In other words *if* u *is the polar line of* X, *then* X *is the polar point of* u. Hence in view of the first theorem of §72

If the vertices of a complete 4-point lie on a conic, each vertex and opposite side of the diagonal triangle are polar point and line with respect to the conic, and dually.

Such a triangle is called *self-polar*.

Further let us consider four points of a conic and the tangents at the points, denoting the points and lines alike by 1 2 3 4. Then it is plain from the figure that the six



vertices of the 4-line and the six sides of the 4-point are polar points and lines with respect to the conic. And (*e. g.*) d_1, the junction of the opposite vertices 12 and 34 of the 4-line, is the polar line of D_1, the intersection of opposite sides 12 and 34 of the 4-point. It follows that

The complete 4-point inscribed in a conic and the complete 4-line touching at the points have a common (self-polar) diagonal triangle.

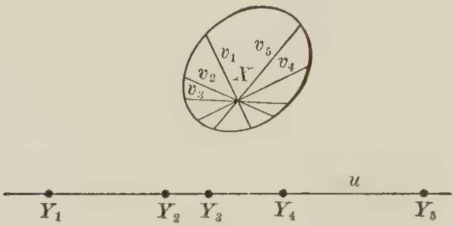
74. From the definitions of §§71, 73 we have at once the *fundamental theorem of poles and polars*

(*I*) *If a point Y run along a line u, its polar line v will turn about a point X, the pole of u. And dually, if a line v turn about a point X, its pole Y will run along a line u, the polar of X.* See figure, next page.

An alternative form of this theorem which is sometimes useful is

(I′) If two points X and Y are such that X lies on the polar of Y, then Y lies on the polar of X.

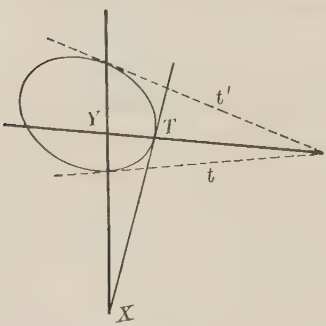
X and Y are then *conjugate points* and their polars u and v *conjugate lines* with respect to the conic.



As applications of these theorems we have

(1) The vertices of a self-polar triangle are conjugate in pairs, and so are the sides. On that account the triangle is sometimes called *self-conjugate.*

(2) *Two conjugate points X and Y are harmonically separated by (a) the end points of the chord determined by them and (b) the tangents at the end points.*



Thus $(XY|t\,t') = -1$. If T is the contact of the tangent from X we have also $(XT|t\,t') = -1$. If now XY is projected into £ we have the special case:

The segment of a tangent to a hyperbola intercepted by the asymptotes is bisected by the point of contact.

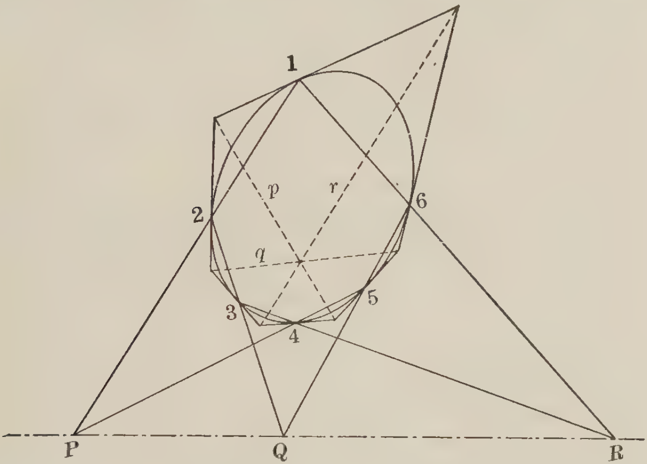
(3) *The polar lines of the points of a range constitute a pencil projective with the range, and dually.*

The first part of the theorem is an immediate corollary

of (I). And since the range and pencil are in (1, 1) correspondence they are also projective (§50).

(4) Brianchon's theorem can be derived from Pascal's by the aid of the polar theory.

Denote the six points and the tangents at the points alike by 1 2 3 4 5 6. Then consulting the figure, it is clear that the intersections P Q R of opposite sides of Pascal's



hexagon are poles of the junctions p q r of opposite vertices of Brianchon's. But P Q R are on a line, hence p q r are on a point (by I).

Incidentally it appears that Brianchon's point and Pascal's line are pole and polar with respect to the conic.

EXERCISES

1. A tangent line and its point of contact are polar line and point with respect to a conic.

2. Given five points, construct the polar line of a point P with respect to the conic determined by the points. Solve the problem when the conic is determined by any five elements as, *e. g.*, four points and the tangent at one of them. Dualize.

3. Construct a triangle self-polar with respect to a conic. How much of the construction is arbitrary, in other words how many constants belong to a conic and a self-polar triangle?

4. If two pairs of opposite sides of a complete 4-point are pairs of conjugate lines with respect to a conic, the third pair are conjugate lines.

5. All conics on four points have a common self-polar triangle. Dualize.

6. The diagonal points of the two simple quadrangles formed by (a) four tangents to a conic and (b) the contacts of the tangents are harmonic pairs on a range. Dualize.

7. Dualize the second theorem of §72.

8. If a point run along a conic, its polar line with respect to a second conic will generate a line conic, and dually.

9. If a point move on a line, its polar lines with respect to two conics will intersect in pairs in points of a conic.

10. The four lines joining any point of a conic to the intersections of two conjugate lines with the conic constitute a harmonic pencil.

11. If the sides of one triangle are polars of the vertices of a second, then the sides of the second are polars of the vertices of the first. The triangles may be called *polar triangles*. What is the polar triangle of an inscribed triangle? Dualize.

12. Polar triangles are perspective, the center and axis of perspection being pole and polar.

13. If a triangle $A\ B\ C$ is inscribed in a conic, the points P, Q in which two sides (as AB and AC) cut any line through the pole A' of the third side are conjugate points with respect to the conic. State and prove the converse and dualize both the theorem and the converse.

14. If two triangles are self-polar with respect to a conic, their vertices lie on a conic and their sides touch a conic.

15. Let a conic be referred to a triangle consisting of two tangents ($x_1 = 0$, $x_3 = 0$) and their chord of contact $x_2 = 0$. And let the lines of the pencil $x_1 - tx_2 = 0$ cut the conic again in points with parameters t. Then show that if lines on $(0, 1, 0)$ meet the conic in points with parameters t and t', $t + t' = 0$.

16. The polar of a point with respect to a circle is perpendicular to the line joining the point to the center of the circle. The product of the distances of a point and its polar line from the center of the circle is constant (equal to the square of the radius).

17. The angle between two lines is equal to the angle at the center of a circle subtended by the poles of the lines with respect to the circle.

75. Reciprocal polars.—We have seen that a conic pairs the points and lines in its plane in a mutual way,—such that every point has a polar line of which in turn it is the pole. The points and lines thus associated with a conic constitute a *polar system* in which polar point and line are *corresponding elements.* Thus to a range corresponds a pencil; to the points of a conic correspond the lines of a conic and generally to a point curve C corresponds a line curve C'.

Regarding more closely the relation between C and C', we observe that collinear points of C go into concurrent lines of C' and vice versa, *i. e.*, the order of one curve is the class of the other. Moreover to a point and its tangent of C correspond a tangent and its contact of C'. Therefore

If a point and line in the united position describe a curve[1] C *of order* n *and class* m, *the polar line and point, which are also in united position, will generate a curve* C' *of class* n *and order* m.

Since the relation between the two curves C and C' is reciprocal, the points and lines of the one corresponding to the lines and points of the other in a polar system, either is called the *polar reciprocal* of the other with respect to the conic which is designated the *auxiliary conic.*

We have thus a phase of the duality already familiar. For any projective theorem concerning points in a polar system can be translated into a reciprocal theorem concerning lines simply by the interchange of the terms pole and polar. This method by which a theorem is inferred from its reciprocal is called *reciprocating with respect to a conic.*[2]

[1] As explained, §12.

[2] The method of reciprocating with respect to a conic was known and practiced by Poncelet some years before the formulation of the principle of duality by Gergonne (1826). Still earlier Brianchon had proved his theorem by the aid of the polar theory. This theory indeed was first developed by Desargues (1639).

There is however one important difference between the aspect of duality as first developed and that presented here. Not only did we first justify duality on analytic grounds, regarding dual properties as alternative representations of the very same abstract truth. But dual figures had no special placing with respect to each other in the plane. Here reciprocal figures as well as reciprocal theorems are derived the one from the other through the mediation of a conic. Thus in the conic and the polar system we have a *geometrical apparatus* for effecting reciprocal operations which serves at the same time as a connecting link between reciprocal figures. Given a figure we can actually construct the reciprocal figure, at least formally, when the two figures have a special position with respect to the conic and therefore to each other. The conic may of course be kept in the background or entirely suppressed when merely passing verbally from one theorem to its reciprocal.

Some Metrical Aspects of the Polar Theory

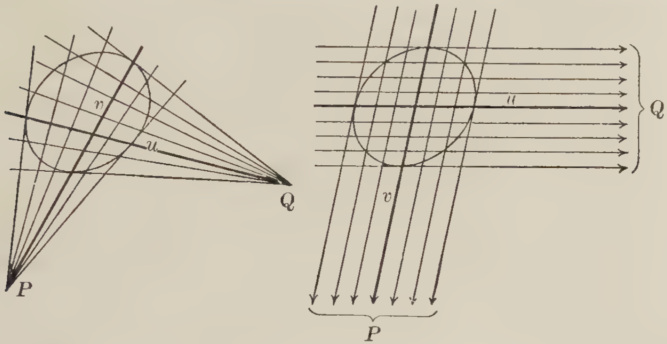
76. Centers and diameters of conics.—If a point P lie at infinity, its conjugate Q with respect to a conic is the midpoint of the chord PQ and lines on P form a parallel pencil. Accordingly the definition (*a*) §71 may be replaced by

The locus of the midpoints of a system of parallel chords of a conic is a line u, the polar line of the point P at infinity in which the chords (produced) intersect.

The line u so defined is called a *diameter* of the conic. If two conjugate points P and Q lie at infinity, their polar lines u and v are designated *conjugate diameters*, each bisecting all chords parallel to the other. In particular of course conjugate diameters bisect each other.

The pole of the line at infinity with respect to the conic is the locus of diameters, a point bisecting all diameters and called therefore the center *of the conic.*

Since a parabola is tangent to the line at infinity its center is the point of tangency. Accordingly *all diameters*



of a parabola, which must pass through the center, are parallel to the axis.

A multitude of metrical theorems are obtained at once by taking into account infinite elements. Some of these will be found in the accompanying exercises.

EXERCISES

1. Define diameter and center corresponding to definitions (*b*), (*c*) and (*d*) of polar line and point §§71, 73.

2. State the metrical form of the second theorem §72 when the point P is at infinity.

3. Two conjugate diameters and the line at infinity form a self-polar triangle.

4. The diagonals of a parallelogram inscribed in a conic are diameters and those of a parallelogram circumscribed to a conic are conjugate diameters.

5. If a parallelogram is circumscribed to a conic, the contacts are the vertices of a parallelogram and if a parallelogram is inscribed in a conic the tangents at the vertices form a parallelogram. In either case the four diagonals are a harmonic pencil.

6. The line (segment) joining the middle point of a chord of a parabola to the pole of the chord is bisected by the curve.

7. Given five lines of a conic construct the center.

8. Any two conjugate diameters of a hyperbola are harmonic conjugates with respect to the asymptotes.

9. Conjugate diameters of a circle are perpendicular.

10. A directrix of a conic is a polar line of the corresponding focus. (Show for example that the tangents at the ends of a focal chord meet on the directrix.)

11. Conics with a common focus and (corresponding) directrix have double contact. What is the chord of contact? Two parabolas with the same focus and directrix coincide.

77. Reciprocating with respect to a circle.—We have seen that the polar reciprocal of a conic C with respect to an auxiliary conic K is a conic C'. In a metrical treatment it is important to know the type of the conic C'. Now the polar of the center O of K is the line at infinity. The infinite points of C' are thus the poles of the tangents of C which pass through O. But these tangents and consequently their polar points are real and distinct, real and coincident or conjugate imaginary according as the point O lies outside, on or inside the conic C. In other words

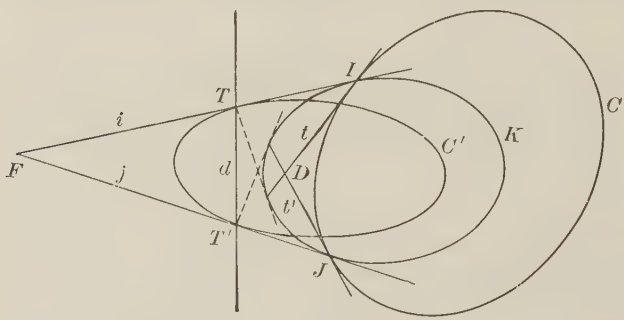


FIG. *a*.

1°. *C', the polar reciprocal of any conic C with respect to an auxiliary conic K, is an ellipse, parabola or hyperbola according as the center of K is inside, on or outside the conic C.*

We shall now examine more specifically the relations of the three conics. By definition the polar lines as to K of points of C will be lines of C' while the polar points of lines of C will be points of C' and vice versa, the relation between C and C' being mutual.

In the figure (a) we designate uniformly polar points and lines with respect to K by corresponding large and small letters. If now I and J are two of the common points of C and K then i and j, meeting at F, will denote the tangents to K at these points. Further let t and t' be tangents to C at I and J respectively and let them meet at D. Then the pole of t is some point T on i and the pole of t' is a point T' on j. Thus the point D lies on the polars of T and T', hence the polar d of D is the line TT'. Moreover C' touches i and j (since they are polar lines of points I, J of C) at T and T' (which are poles of lines t, t' of C). Therefore

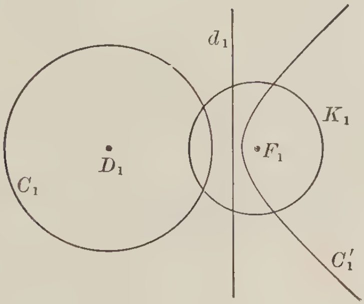


FIG. b.

2°. *C' is a conic with respect to which F (the pole of f as to K) and d (the polar of D as to K) are polar point and line.*

Now suppose the whole figure (a) projected so that I and J go into the circular points. Then (Fig. b) C and K become circles C_1, K_1 with centers D_1 and F_1 (since these

points are poles of (the new) $\mathcal{L}$ as to C_1 and K_1. And since a focus of a conic is an intersection of a pair of tangents from I and J (§35, Ex. 16) while a directrix is the polar line of the corresponding focus (§76, Ex. 10), C' is projected into a conic C_1' with F_1 as focus and d_1 as directrix. Thus we have proved

3°. *The polar reciprocal of a circle C with respect to an auxiliary circle K is a conic C' whose focus is the center of K and whose directrix is the polar with respect to K of the center of C.* And conversely, *the polar reciprocal of any conic C' with respect to a circle K with center at a focus of C' is a circle C whose center is the pole as to K of the directrix of C'.*

The type of C' is determined by the criterion 1°. Further when the auxiliary conic is a circle we have the basic theorems (§74, Exs. 16, 17).

4°. *The product of the distances of a point and its polar line from the center of a circle is constant and equal to the square of the radius.*

5°. *The angle between two lines is equal to the angle at the center of the circle subtended by the poles of the lines.*

In order to facilitate the application of the method let us examine anew the machinery to be used. We shall write in parallel columns corresponding (reciprocal) items, O denoting the center of the auxiliary circle.

point	line (polar of the point)
line	point (pole of the line)
point and line on it	line and point on it
two points and their junction	two lines and their intersection
point locus	line locus (envelope)
circle C	conic C' with O as focus
center of circle C	directrix of conic C'

point of C	tangent of C'
tangent of C	point of C'
tangent t and contact P of C	point T of C' and tangent p with contact T
concentric circles	conics with same focus and directrix.

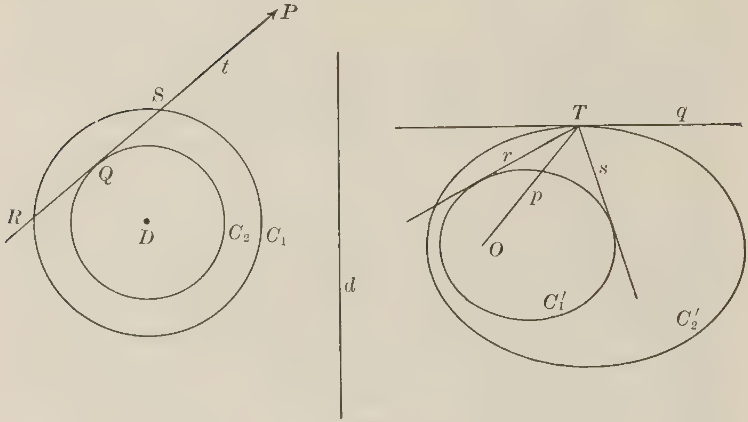
The student is referred again to the remarks on dualizing, §45. In solving problems it is recommended that he first draw a figure illustrating the theorem whose reciprocal is in question. Then he should construct the reciprocal figure step by step making sure that the reciprocal of every essential element of the first figure is represented. Finally he will describe the corresponding property of these reciprocal elements in a *reciprocal theorem.*

In practice the auxiliary circle may be dispensed with and the process called *reciprocating with respect to a point,* the center of the auxiliary circle. Thus if O is the center and k the radius of the auxiliary circle and OP and OT the distances from the center of a point P and its polar line, the reciprocal of a conic with respect to the point O may be defined as the locus of a point P which moves subject to the condition $OP \cdot OT = k^2$.

Example. Reciprocate with respect to a point O the theorem: *If two circles are concentric, a chord of one which is tangent to the other is bisected at the point of contact.*

Let D be the center of the two circles C_1 and C_2 and let t be a line cutting C_1 at R, S and touching C_2 at Q. The infinitely distant point of t is denoted by P. We have then the reciprocal items

circles C_1 and C_2	conics C_1' and C_2' with common focus O and directrix d



tangent t of C_2	point T of C_2'
Q, contact of t on C_2	q, tangent to C_2' at T
R, S, common points of t and C_1	tangents r, s of C_1' from T
P, intersection of t and $\mathcal{L}$	p, junction of T and O
$(PQ \mid RS) = -1$	$(pq \mid rs) = -1$

The reciprocal theorem is accordingly: *If two conics have a common focus and directrix, the tangent and focal chord of any point on one divide harmonically the tangents from the point to the other.*

In this example theorem 3° enabled us to derive a property of conics from a simple property of circles. The converse of the theorem may also be used effectively in the solution of problems on conics, for upon reciprocation the question to be resolved is frequently obvious. Thus suppose we are required to prove the theorem just deduced. Reciprocating with respect to the focus the figure on the right we obtain the figure on the left and the original theorem which is obvious.

EXERCISES

In Exs. 2–9, K refers to the auxiliary conic while C and C' denote a pair of conics reciprocal with respect to K.

1. What is the reciprocal of a conic with respect to itself?

2. The common points of C' and K are contacts (on K) of common tangents to C and K.

3. The common lines of C' and K are the tangents to K at the common points of C and K.

4. Common points and common lines of C and C' are poles and polars with respect to K.

5. There are four points on a conic C whose polar lines as to another conic K touch C, *viz.*, the intersections of C and its reciprocal C'. Dualize.

6. Draw the simple 4-lines touching C and K respectively at their common points. Then C' touches the sides of the 4-line circumscribing K at the poles of the sides of the 4-line circumscribing C.

7. Show that if C touches K, C' touches K at the same point. Hence the reciprocal of any member of a system of double contact conics with respect to any other member of the system is a member of the system.

8. What are the reciprocals of the following with respect to a conic: (*a*) asymptote, (*b*) parallel lines, (*c*) center of a conic, (*d*) parallel tangents of a conic, (*e*) concentric conics, (*f*) a pair of conjugate diameters of a conic, (*g*) foci of a conic?

9. If C and K have the same lines for principal axes (perpendicular diameters), C' will have the lines for principal axes also. Construct a figure for this case.

10. What is the reciprocal of a hyperbola with respect to the conjugate hyperbola? Of a circle with respect to its center?

11. What is the reciprocal of the circular points with respect to a circle? the foci of a conic? a diameter of a circle?

12. Prove directly that when a circle is reciprocated with respect to a circle (Fig. *b*) the eccentricity of the resulting conic is the ratio of the lengths of the segment of centers (D_1F_1) to the radius of C_1, *i. e.*, the shape of the reciprocal is independent of the size of the auxiliary circle. (Make use of the relation 4°.)

13. Show that Ex. 12 proves 1° for C and K circles.

14. Trace the motion of the generating point on the hyperbola (Fig. *b*) as the variable tangent moves around the original circle. Draw the other branch of the hyperbola.

15. Draw Fig. b when F_1 is (a) on, (b) inside the original circle.

16. In Fig. b the line of centers is by symmetry the axis of the reciprocal conic. Then by 4° and Ex. 12 find the elements (a, b, c, e, p) of C_1' when $D_1F_1 = 3$, radius of C_1 is 5 and radius of K_1 is 10.

78. We shall now explain how the foregoing principles can be employed in a slightly different manner to obtain certain quasi-projective theorems concerning conics from elementary properties of the circle.[1] We use the same apparatus as before, an auxiliary circle K, a circle C whose reciprocal with respect to K is a conic C' having the center of K as focus. We use however the theorem which is fundamental for the method:

The angle between two lines connected with C is equal to the angle at the focus of C' subtended by the poles of the lines (5°, §77).

We begin with a familiar theorem usually involving a circle and a pair of lines which meet at a definite angle. Proceeding as in §77 we draw an illustrative figure which we reciprocate in every essential detail, making use of the table of corresponding terms. But here the similarity ceases and we apply to the lines the fundamental theorem. The derived theorem is accordingly *not* the reciprocal of the first. For in genuine reciprocation, as in dualizing, we merely interchange corresponding terms (pole and polar, etc.). Here indeed we reciprocate the figure associated with a theorem but instead of stating the reciprocal theorem we infer a new theorem by utilizing a polar property of the circle.

The method is noteworthy since it enables us to translate into rather general theorems properties involving angular magnitudes.

Likewise we may transform certain theorems involving

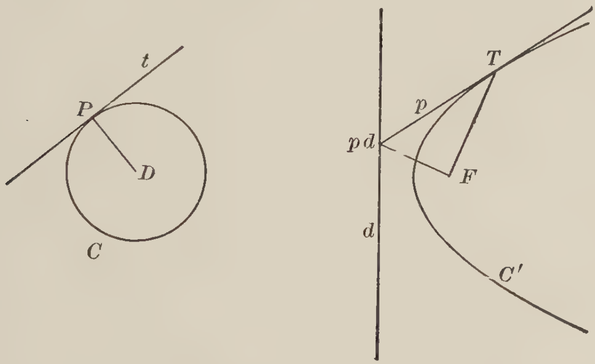
[1] We say quasi-projective for while the theorems are stated in metric form they are valid alike for all types of conics. Most of them in fact can be given a projective statement.

distances by a similar process, using however 4° of §77 as the fundamental theorem (see below, Exs. 21–23).

We shall now illustrate the method by one or two examples.[1]

Example. To deduce a theorem for conics from

(1) *A tangent to a circle is perpendicular to the radius drawn to the point of contact.*



Let D be the center of the circle C, t the tangent and P the point of contact. Taking F as the center of reciprocation, we have the following correspondence

circle C	conic C' with F as focus
tangent t and contact P of C	point T and tangent p of C'
center D	directrix d
line PD	point pd (intersection of tangent with directrix).

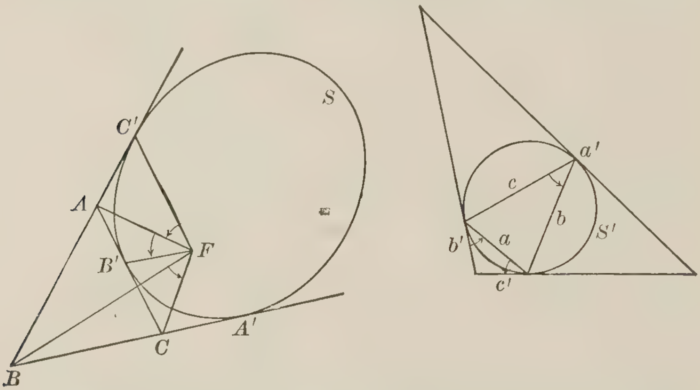
But PD is perpendicular to t. Hence applying the fundamental theorem, the lines joining pd and T (the poles of PD and t) to F are perpendicular and our new theorem is

[1] While the figures for the examples in this and the preceding section are intended to be accurate in the sense that the figures right and left are reciprocals of each other, the two parts are not in proper relative position, *i. e.*, the center of reciprocation is displaced.

10

(1′) *Any point of a conic and the intersection of the tangent at the point with the directrix subtend at the focus a right angle.*

To apply the method to the solution of a problem, let us prove the theorem: *If the sides BC, CA, AB, of a triangle touch a conic at the points A', B', C' respectively, then at either focus BC, $B'A$, $C'A$ subtend equal angles.*



Let F be a focus of the conic S in question. We are to show that the angles BFC, $B'FA$, $C'FA$ are equal. Reciprocating with respect to F we obtain the reciprocal correspondence

conic S	circle S'
vertices A, B, C of triangle circumscribed to S	sides a, b, c of triangle inscribed in S'
contacts A', B', C' of sides of triangle ABC	tangents a', b', c' at vertices of triangle abc
angle subtended at F by $\begin{cases} BC \\ B'A \\ C'A \end{cases}$	angle formed by polars $\begin{cases} b, c \\ b', a \\ c', a \end{cases}$

(by the fundamental theorem)

Now the angles on the right are manifestly equal, being measured by the same arc, Q.E.D. Likewise the rest of the theorem is proved.

EXERCISES

Derive theorems for a conic from the following theorems of the circle by the methods of this section.

1. Two tangents to a circle make equal angles with their chord of contact.

2. Two tangents to a circle make equal angles with the diameter through their common point.

3. Any line is perpendicular to the line joining its pole to the center.

4. Parallel tangents to a circle touch it at the ends of a diameter. Hence prove that a directrix is polar of a focus.

5. The locus of the point of intersection of two tangents to a circle which meet at a constant angle is a concentric circle.

6. The envelope of the chord of contact of the tangents in Ex. 5 is a concentric circle.

7. The angle inscribed in an arc of a circle is constant. If the arc is a semicircle the angle is a right angle.

8. If the base and vertex angle of a triangle are fixed, the vertex moves on a circle through the extremities of the base. If the vertex angle is right the circle has the base as diameter. What is the directrix of the corresponding conic?

9. If from a fixed point tangents be drawn to a family of concentric circles, the locus of the points of contact is a circle through the fixed point and the common center. State the problem when the fixed point is the center of reciprocation.

10. The locus of the intersection of perpendicular tangents to an ellipse or hyperbola is a circle.

11. If through a fixed point on a circle a variable pair of perpendicular chords be drawn, the locus of the line joining their extremities is the center.

12. The envelope of a chord of a circle which subtends a constant angle at a fixed point on the curve is a concentric circle.

Prove the following theorems by reciprocating with respect to a point:

13. The reciprocal of a parabola with respect to a point on the directrix is a rectangular hyperbola.

14. If a rectangular hyperbola is reciprocated with respect to a point O, the tangents from O to the reciprocal conic are perpendicular.

15. The orthocenter O of a triangle circumscribing a parabola lies on the directrix. (Reciprocate with respect to O. If O is on the directrix, the tangents from O to the parabola are perpendicular, Ex. 11.)

16. The orthocenter O of a triangle circumscribing a rectangular hyperbola is on the curve. (Reciprocate with respect to O and apply the illustrative example.)

17. Two conics C_1, and C_2 with directrices d_1, d_2 have a common focus F. Then the polar of d_1 with respect to C_2 and the polar of d_2 with respect to C_1 are two points collinear with F.

18. With any vertex of a quadrangle as focus one conic can be drawn inscribed and four conics can be drawn circumscribed to the triangle formed by the other three vertices.

19. The locus of a variable chord PQ which subtends a right angle at a point O on a conic is a point (on the normal at O).

20. What is the locus of the chord PQ (Ex. 19) which subtends a constant angle at a point on the conic?

By reciprocating with respect to a circle and then applying 4°, §77, prove the following theorems, those on the right from those on the left:

21. The sum of the perpendiculars from the point of reciprocation (between the tangents) to two parallel tangents to a circle is constant,—equal to the diameter of the circle.

21′. The sum of the reciprocals of the segments of any focal chord of an ellipse is constant,—equal to four times the reciprocal of the focal chord perpendicular to the major axis. Hence this chord is the same for all ellipses which are reciprocals of equal circles with respect to any point.

22. The product of the segments of any chord of a circle through the point of reciprocation is constant.

22′. The product of the perpendiculars from the focus of an ellipse upon two parallel tangents is constant.

23. The sum of the focal radii drawn to contacts of parallel tangents of an ellipse is constant,—equal to the length of the major axis. Prove.

23′. The sum of the reciprocals of perpendiculars from any point within a circle to two tangents whose chord of contact passes through the point is constant.

79. The method of projection.—The student has had no little experience in passing from projective to metric properties, a transition always accomplished through the mediation of the absolute. But he has had slight practice in the reverse procedure,—that of inferring a projective theorem from one of its metric versions. For this purpose however there is now available a more adequate vocabulary of projective-metric terms by the help of which many simple metric properties can be translated into more general projective theorems.

The student should collect for reference the projective equivalents of such terms as parallel lines, perpendicular lines, angle, center of a segment, ellipse, parabola, hyperbola, circle, rectangular hyperbola, asymptote, focus, directrix, center and diameter of conic, conjugate diameters, concentric circles, parallelogram, confocal conics, etc.

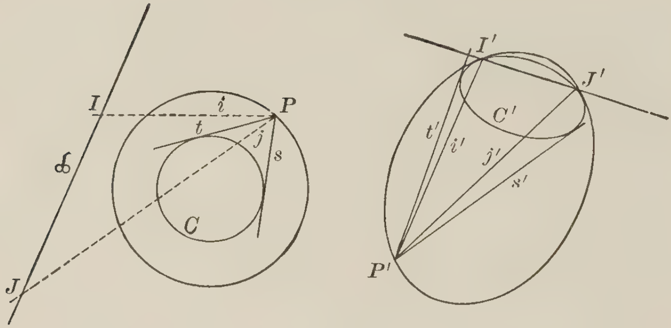
It will also be recalled that projection is a point-point transformation which preserves double ratios and such relationships as incidence, collinearity, concurrency, tangency, the relationship of pole and polar; that under projection a circle (or any conic) goes into a conic but magnitudes change.

The method now will be briefly exemplified.

Example 1. *The locus of intersection of two tangents to a circle which meet at a constant angle is a concentric circle.*

Let C be the circle, t and s tangents meeting at P to form a constant angle and denote by i, j the circular rays from P. If we project so that the circular points I, J go into finite points I', J', we have as corresponding parts of the two figures

circle C	conic C' on two fixed points I', J'
tangents t, s	tangents t', s' meeting at P'



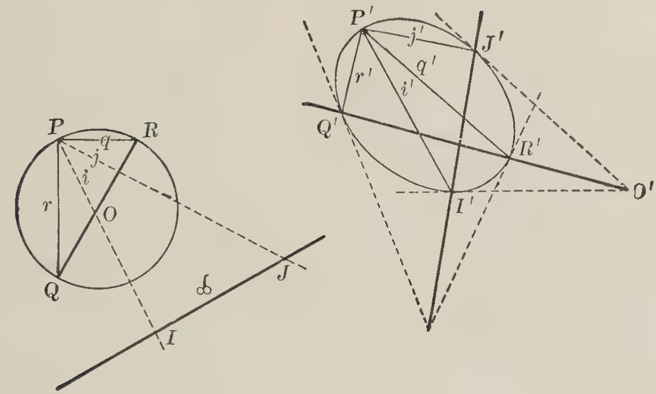
lines i, j	lines i', j' from P' to I', J'
Now angle P is constant, *i. e.* $(ts \mid ij) =$ constant	hence $(t's' \mid i'j') =$ constant
The locus of P is a concentric circle with contacts at I, J.	hence the locus of P' is a conic touching C' at I', J'.

Therefore: *The locus of the center of the pencil of constant double ratio, formed by two variable tangents to a conic and lines to the extremities of a fixed chord, is a conic touching the first at the extremities of the fixed chord.*

Example 2. *An angle inscribed in a semicircle is a right angle.*

Let P be any point and QR any diameter of a circle with center O and denote by q, r the sides of the angle RPQ and by i, j the circular rays from P. Now the pole of QR is at infinity, *i. e.*, QR and $\mathcal{L}$ are conjugate polar lines. Projecting the figure, we have the following correspondents

circle	conic
point P on circle	point P' on conic
$\mathcal{L}$ (cutting circle at I, J)	line cutting conic at I', J'
center O	pole O' of $I'J'$
QR (conjugate to $\mathcal{L}$)	$Q'R'$, conjugate to $I'J'$



circular rays i, j	lines i', j', joining P' to I', J'
sides q, r	lines q', r' joining P' to Q', R'
But $(qr \mid ij) = -1$	hence $(q'r' \mid i'j') = -1$.

And the projective theorem is

Lines joining any point of a conic to the extremities of conjugate chords form a harmonic pencil.

EXERCISES

Derive projective theorems corresponding to the following. Show that the same results can be obtained by reciprocating with respect to a conic and then dualizing.

1. If two circles are concentric, a chord of one tangent to the other is bisected by the point of contact.

2. If two pairs of opposite sides of a hexagon inscribed in a circle are parallel, the other pair of opposite sides are parallel.

3. A tangent to one of three concentric circles cuts the other two in four points whose double ratio is constant.

4. A tangent to a circle is perpendicular to the radius drawn to the point of contact.

5. The locus of the intersection of perpendicular tangents to a parabola is the directrix.

6. Any line is perpendicular to the line joining its pole to the center of the circle.

7. The envelope of the chord of contact of tangents to a circle which cut at a constant angle is a concentric circle.

8. The locus of the center of a circle on a fixed point and touching a fixed line is a parabola having the point and line for focus and directrix.

9. The locus of the center of a circle touching two given circles is a hyperbola having the centers of the two given circles as foci.

10. The locus of the intersection of perpendicular tangents to a central conic is a circle, concentric with the conic.

11. The envelope of a chord of a conic which subtends a constant angle at the focus is a conic with the same focus and directrix.

12. The diagonals of a parallelogram bisect each other.

13. The locus of a point which divides a system of parallel chords of a circle in a constant ratio is an ellipse having double contact with the circle.

14. The diagonals of a parallelogram inscribed in a circle are bisected at the center. The diagonals of the parallelogram formed by tangents at the vertices of the first meet at the center and bisect the angles between the other diagonals.

15. The envelope of a chord of a circle which subtends a constant angle at a fixed point on the curve is a concentric circle.

16. The perpendicular bisectors of the sides of a triangle meet at the center of the circumscribed circle.

17. The lines from the vertices of a triangle perpendicular to the opposite sides meet in a point.

18. The pair of "exterior" as well as the pair of "interior" tangents to two circles meet on the line of centers.

19. If a triangle is inscribed in a circle, the perpendiculars from any point on the circle meet the sides of the triangle in three collinear points.

20. The two pairs of lines joining the ends of parallel diameters of two circles meet on the line of centers.

Prove the following theorems by projecting the figures into simple metrical forms,—projecting a line into the line at infinity, two points into the circular points, a conic into a circle, etc.

21. If two conics have double contact at P, Q, then any third conic on P and Q cuts each of the others in an additional pair of points.

The junctions of these two pairs meet on PQ. (Project P and Q into the circular points.)

22. A variable conic is drawn to touch two fixed lines and to pass through two fixed points P, Q. The locus of the intersection of tangents to the conic at P and Q is a pair of lines harmonic with the fixed lines.

23. Two conics, C_1, C_2, have double contact and O is the pole of the chord of contact. From two points P, Q on C_1 tangents are drawn to C_2 forming a quadrilateral. If D is the pole of PQ with respect to C_2 show that two of the diagonals of the quadrilateral meet at D and one of them goes through O.

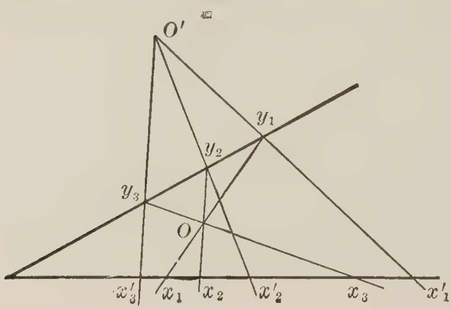
24. Let D_1, D_2 be the diagonal points and d_1, d_2 meeting at O, the diagonal lines of a simple quadrangle and let P, Q be a pair of points collinear with D_1, D_2 and harmonic both with D_1, D_2 and d_1, d_2. Then two conics may be drawn, one on the vertices and one on the sides of the quadrangle-quadrilateral, having double contact at P, Q and with O as the common pole of the chord of contact.

25. Two conics have double contact at P, Q and two tangents t, t' to one conic cut the other in the vertices of a 4-point of which t, t' are one pair of opposite sides. A second pair of opposite sides together with the chord of contact of t and t' are concurrent in a (diagonal) point lying on PQ. The other two diagonal points are collinear with the pole of PQ with respect to the conics. A conic can be drawn having double contact with the others at P, Q and touching one pair of opposite sides of the quadrangle at the points cut out by the chord of contact of t and t'.

CHAPTER VII

COLLINEATIONS AND INVOLUTIONS IN ONE DIMENSION

80. Geometric definition of collineation.—We have hitherto considered projective ranges as situated on distinct lines. We may however think of them as superposed on the same line. Two such ranges may be obtained as follows. From a center O project the points x_i of a



range x into the points y_i of a second range y. Then from a center O' project the points y_i back into points x_i' of the first range. It follows of course that $x \overline{\wedge}\, y \overline{\wedge}\, x'$. These two projections successively applied effect a rearrangement of the points of the range x which may be described as a (1, 1) projective correspondence and which is called a collineation.

Dually by two successive perspectivities with distinct axes we may set up a (1, 1) correspondence between the lines of a pencil which is also called a collineation. Or formally

A collineation in one dimension is a projective correspondence between the elements of a one-dimensional form.

A collineation is thus a correspondence between elements of the *same kind* (point-point or line-line) in the *same* form, whereas in a general projectivity the correspondence may be between points of different ranges, lines of different pencils or points of a range and lines of a pencil.

81. The fixed points of a collineation.—The properties of a collineation are most conveniently studied by means of the analytic expression which defines it. We shall deduce some of these properties, stating the results however for point transformations only and leaving the dual as usual to be supplied by the student.

We have seen (§50) that if x and x' are coördinates of corresponding points in a collineation along the line, then *the collineation is represented by the bilinear equation*

$$x' = \frac{ax + b}{cx + d}, \quad ad - bc \neq 0. \tag{1}$$

Further, *a collineation is completely determined when three pairs of corresponding points are given.*

Equation (1) is to be regarded as the operation or transformation which sends the point x,—the point itself and not its coördinate,—into the point x'. The superiority of the analytic over the geometric definition is now apparent for in the one case a collineation is given by a single analytic operation while in the other two perspectivities were required.[1] The student should not lose sight of the geometric definition however for it will prove serviceable. Indeed it will be profitable to establish succeeding theorems by geometric methods.

[1] In the analytic representation it is not necessary to take account of the geometrical machinery which carries the point x into the point x'. It suffices to know that in virtue of (1) x' is determined when x is known. The geometric definition holds the advantage when corresponding points are actually to be constructed.

The first question of importance is: Are there any points of the range which are fixed under the transformation (1), *i. e.*, are there any *self-corresponding* points in a collineation? Since x and x' are coördinates of corresponding points, we shall have fixed points when $x = x'$. Making this substitution we obtain

$$cx^2 + (d - a)x - b = 0, \tag{2}$$

a quadratic in x. *Hence in a collineation of the line there are always two fixed or self-corresponding points.*

Cor. *If a collineation have more than two fixed points, all its points are fixed*, for a quadratic with three roots vanishes identically (§5). We should have then $b = c = 0$, $d = a$ so that the collineation becomes $x' = x$, which is called the *identical collineation* or the *identity*, denoted by 1.

To be given a fixed point is equivalent to being given a pair of corresponding points. Hence we may say

A collineation is determined by (*a*) *the fixed points and a pair of corresponding points or* (*b*) *by one fixed point and two pairs of corresponding points.*

The fixed points of a collineation alone suffice to determine only two of the three constants. There are thus ∞^1 collineations of the line with identical fixed points.

If the fixed points are f_1, f_2 and k is a constant the collineation may be written

$$\frac{x' - f_1}{x' - f_2} = k\frac{x - f_1}{x - f_2}. \tag{3}$$

For (3) is the equation of a collineation since it comes under the general class (1). Furthermore when $x = f_1$, $x' = f_1$ and when $x = f_2$, $x' = f_2$.

Now writing (3) in the form

$$k = \frac{(x' - f_1)(x - f_2)}{(x' - f_2)(x - f_1)} \equiv (x'x|f_1f_2), \tag{4}$$

it appears that *the double ratio of the fixed points and any pair of corresponding points of a collineation is constant.*

This theorem may be regarded as implying a *third definition* of a collineation.

82. Types and standard forms of collineations.—Collineations may be classified according to the nature of the roots of the quadratic giving the fixed points ((2) §81). We have thus the following varieties with names as indicated.[1]

fixed points	*collineation*
1°. real and distinct	hyperbolic
2°. real and coincident	parabolic
3°. conjugate imaginary	elliptic.

The equation of a collineation can be materially simplified by a proper selection of the fixed points. Thus if 0 and ∞ are taken as fixed points we must have (from (2) §81) $b = c = 0$, when the collineation may be written

$$x' = kx. \tag{1}$$

Equation (1) which is the simplest form to which the general collineation can be reduced is called on that account the *canonical* form.

Likewise if both the fixed points are ∞, $c = 0$ and $d = a$ and the collineation takes the form

$$x' = x + k, \tag{2}$$

which is the canonical form of a parabolic collineation. The collineation is however special.

It remains to consider a case which has been excluded hitherto, *viz.*, when $ad - bc = 0$. Suppose for the moment

[1] These names have no essential connection with the various conics, they are only suggested by the relation of the conics to the line at infinity.

The classification here given is not altogether projective, for in a strict projective classification we should not distinguish real and imaginary. It is however a projective specialization for a collineation to be parabolic or singular.

that none of the coefficients is zero, then $a/b = c/d$ and $a/c = b/d$. If now the collineation be written in the form

$$x' = \frac{b}{d} \frac{a/b\ x + 1}{c/d\ x + 1} \tag{3}$$

we distinguish two cases

I. $x \neq -b/a (= -d/c)$ and II. $x = -b/a (= -d/c)$

Under I, (3) becomes $x' = b/d$ and is the same for every x. While under II, $x' = 0/0$. The results are the same, as the student may verify, when some of the coefficients are zero. The collineation is said to be *singular* or *degenerate*. Or we may summarize

4°. *When* $ad - bc = 0$ *the collineation is degenerate. Then every point of the range except one* $(-b/a \equiv -d/c)$ *corresponds to the same point* $(b/d \equiv a/c)$, *while the exceptional point corresponds to an arbitrary point.*

EXERCISES

1. Find the fixed points of the collineations

$$x' = \frac{x+1}{x-1}, \qquad x' = \frac{x-i}{x+i}, \qquad x' = \frac{2x-1}{3x+2}.$$

What is the type of each collineation?

2. Write collineations whose fixed points are both 0, both 1.

3. Determine the collineations which have the fixed points ± 1; ω, ω^2; 2, -5, and in each of which $x = 0$ corresponds to $x' = \infty$.

4. Find the collineation which sends respectively 0, 1, ∞ into 1, ∞, 0; 1, ∞, 0 into1, ω, ω^2; 1, 2, 3 into 0, ∞, -1. Find the fixed points of each.

5. Determine the collineation with fixed point 0 and which sends 1, ∞ into -1, 2; fixed point ∞ and which sends 1, -1 into i, $-i$; fixed point 0 and which sends 1, -1 into -1, 1. What is the other fixed point of each collineation?

6. Write the condition that the general collineation be parabolic, elliptic, hyperbolic.

7. Construct the fixed points of a collineation which is defined geometrically. When will they coincide? Show that the double

ratio of the fixed points and a pair of corresponding points, is equal to that of the centers and the axes of the perspectivities and hence constant. Dualize.

8. The intersections of corresponding lines of two projective pencils with any line are corresponding points in a collineation. Where are the fixed points? When is the collineation singular? Locate the two exceptional points.

9. Hence given a conic, construct a collineation of each type.

10. What modification is necessary in Ex. 8 if the two pencils are perspective?

11. Dualize Exs. 8, 9, 10.

12. A thermometer is graduated in both the Centigrade and Fahrenheit scales. Obtain the relation between the readings on the two scales. For what temperatures are the readings the same? Is this an alias or an alibi?

83. Involution.—We defined a collineation geometrically as the resultant of two perspectivities. These perspectivities must always be performed in the same order, using, *e. g.*, first O and then O' as a center. A few trials will convince the student that the collineation which sends x into x' will not in general bring x' back into x. That is, while x may be said to correspond to x' in the collineation, x' does not in general correspond to x.[1] This is also analytically evident. To fix the ideas let us consider the collineation

$$x' = \frac{x - 1}{2x - 6} \tag{1}$$

which we interpret as the operation which carries x into x'. Then when $x = 5, x' = 1$, but when $x = 1, x' = 0$. In other words the point 5 corresponds to 1 but 1 does not correspond to 5. Now it might happen that the collineation which transforms x into x' would at the same time transform x' back into x. Such a collineation is called an

[1] A similar remark applies to any general (1, 1) correspondence. A (1, 1) correspondence merely means that each element say x_i of one system corresponds to some element y_i of another and conversely but not that the y which corresponds to an x has for a correspondent the original x.

involution. More specifically since it deals with pairs of points it is said to be of *order two* or *quadratic.*[1] And since either point of a pair determines the other the involution in question is denoted by $I_{1,1}$. We have then a first

1°. **Definition.**—*A quadratic involution* $I_{1,1}$ *is a collineation of period two.*

To find the condition that the general collineation shall be an involution. It is evident from what has been said that the equation must be the same when the rôles of x and x' are interchanged, *i. e.*, we must have simultaneously

$$x' = \frac{ax + b}{cx + d} \qquad \text{and} \qquad x = \frac{ax' + b}{cx' + d} \tag{2}$$

or

$$\begin{aligned} cxx' + dx' - ax - b &= 0 \\ cxx' - ax' + dx - b &= 0. \end{aligned} \tag{3}$$

Whence by subtraction $(d + a)(x' - x) = 0$. Therefore either $x' = x$, in which case we have the identical collineaton, or

$$d + a = 0, \tag{4}$$

which is the condition sought.

Or we may proceed as follows. Solving for x the transformation T, we obtain the transformation denoted by T^{-1}:

$$T: \qquad x' = \frac{ax + b}{cx + d}, T^{-1}: \quad x = \frac{-dx' + b}{cx' - a} \tag{5}$$

If now T is the operation which carries x into x', then T^{-1} is the operation which recovers x from x'. T^{-1} is called therefore the *inverse* of T.

An involution is thus a transformation which is identical with its inverse. This requires

$$\frac{ax + b}{cx + d} \equiv \frac{-dx + b}{cx - a},$$

[1] Involution in this chapter always refers to a quadratic involution.

whence, equating coefficients of like powers (§5)

$$a = -d, \text{ or } a + d = 0,$$

the same condition as before. Unless indeed $b = c = 0$ when $a = d$ is a solution but then T reduces to the identity, a trivial case.

Thus the equation of the involution is

$$x' = \frac{ax + b}{cx - a}. \tag{6}$$

It appears then that

2°. *An involution is defined analytically by a symmetrical equation of the form*

$$Axx' + B(x + x') + C = 0 \tag{7}$$

where x and x' are corresponding points.

84. Further classification of collineations.—It will be convenient to use homogeneous coördinates when the equation of the collineation ((1) §81) is replaced by the two equations

$$\begin{aligned} \rho x_1' &= ax_1 + bx_2 \\ \rho x_2' &= cx_1 + dx_2 \end{aligned} \tag{1}$$

where ρ is a factor of proportionality. To find the fixed points we set $x_1' = x_1$ and $x_2' = x_2$, obtaining the two linear equations homogeneous in two variables

$$\begin{aligned} (a - \rho)x_1 + bx_2 &= 0 \\ cx_1 + (d - \rho)x_2 &= 0. \end{aligned} \tag{2}$$

The condition that these be consistent, found by eliminating the x's, is

$$\begin{vmatrix} a - \rho & b \\ c & d - \rho \end{vmatrix} = 0,$$

or

$$\rho^2 - (a + d)\rho + (ad - bc) = 0, \tag{3}$$

which we shall write

$$\rho^2 - I_1\rho + I_2 = 0. \tag{4}$$

Corresponding to each value $\rho_{1,2}$ of ρ in this equation is a fixed point of the linear transformation (1). The coefficients in (4) are called *invariants* of the transformation, for a reason which will appear later. By means of them collineations are readily characterized. The values of the invariants and the meaning of certain conditions imposed on them are

$$I_1 = a + d, \qquad I_2 = ad - bc$$

Invariant relation	*effect on ρ's*	*collineation is*
$I_1 = 0$	$\rho_1 + \rho_2 = 0$	an involution
$I_2 = 0$	$\rho_1 = 0$	degenerate
$I_1^2 - 4I_2 = 0$	$\rho_1 - \rho_2 = 0$	parabolic
$I_1 = I_2 = 0$	$\rho_1 = \rho_2 = 0$	degenerate $I_{1,1}$.

85. We shall now explain *why the functions of the preceding paragraph are called invariants.* Let $T\colon x' = \dfrac{ax + b}{cx + d}$ represent a collineation, *i. e.*, an alibi, which changes the position of the point from x to x'. And let $S\colon y = \dfrac{\alpha x + \beta}{\gamma x + \delta}$ denote a transformation of coördinates, *i. e.*, an alias, which changes the coördinate x of one system into the coördinate y of a second system. Further by ${}_{x'}T_x$, ${}_yS_x$, etc. indicate the effect of a transformation, namely that T transforms x into x', S transforms x into y, etc. The result of applying first S and then T is called the *resultant* or *product* of S and T and is denoted by TS.[1]

The product STS^{-1} is called the *transform* of T by S. The effect of this product on y for example is ${}_{y'}S_{x'}T_xS_y^{-1}$

[1] It should be noted that the process of forming a product as here defined is not in general commutative.

i. e., the net result is to transform y into y'. Calculating the product we have

$$x = \frac{-\delta y + \beta}{\gamma y - \alpha} \equiv S^{-1} \tag{1}$$

$$x' = \frac{ax + b}{cx + d} = \frac{(-\delta a + \gamma b)y + \beta a - \alpha b}{(-\delta c + \gamma d)y + \beta c - \alpha d} \text{ (from (1))} \equiv TS^{-1} \tag{2}$$

$$y' = \frac{\alpha x' + \beta}{\gamma x' + \delta} = \frac{a'y + b'}{c'y + d'} \text{ (from (2))} \equiv STS^{-1} \tag{3}$$

where

$$a' = -\delta(\alpha a + \beta c) + \gamma(\alpha b + \beta d),\ b' = \beta(\alpha a + \beta c) - \alpha(\alpha b + \beta d)$$

$$d' = \beta(\gamma a + \delta c) - \alpha(\gamma b + \delta d),\ c' = -\delta(\gamma a + \delta c) + \gamma(\gamma b + \delta d).$$

Whence

$$\begin{aligned} (a' + d') &= -(\alpha\delta - \beta\gamma)(a + d) \text{ or } I_1' = -kI_1 \\ a'd' - b'c' &= (\alpha\delta - \beta\gamma)^2(ad - bc) \text{ or } I_2' = k^2 I_2 \\ \frac{(a' + d')^2}{(a'd' - b'c')} &= \frac{(a + d)^2}{(ad - bc)} \text{ or } \frac{I_1'^2}{I_2'} = \frac{I_1^2}{I_2}. \end{aligned} \tag{4}$$

Hence the functions $a + d$ and $ad - bc$ of the coefficients in T are to a factor[1] equal to the corresponding functions formed for the transform of T by S. In other words they are invariant when T is transformed by S. The third function in (4) which is identical in the original and transformed collineation is called for that reason an *absolute invariant.*

Let us now examine a little more closely the complete effect of the transform STS^{-1} when applied to the points y of the range. First the coördinate y of a point is changed to the coördinate x, then the point x is moved to the new

[1] Note that the factor in each case is a power of the determinant of S.

position x', finally the coördinate x' is changed back into y' of the original system. Thus whereas T merely causes a certain shifting of the points of the range, the transform determines the same shifting while at the same time changing the coördinate system.

An invariant relation therefore is one which is independent of the coördinate system in which the collineation is represented.[1] Thus if a collineation is involutory or parabolic so also is its transform. An elliptic collineation may however be transformed into a hyperbolic. *Invariant relations in short are projective relations.*

In the foregoing we have supposed that T defined a collineation of points and S a collineation of coördinates. But obviously the algebra is identical, in particular equations (4) are valid, if both represent transformations of the *same* kind. We should then have other interpretations for the transform STS^{-1} but we should still define an invariant of T as a function of the coefficients which differed at most by a constant factor from the corresponding function of the coefficients in the transform. And any projective property of T would persist in the transform as before.

86. We return now to a consideration of the involution, writing its equation in the form

$$Axx' + B(x + x') + C = 0. \tag{1}$$

Since (1) is a two-parameter family of transformations, it can be made to satisfy two conditions. Hence an involution is determined by (*a*) two pairs of corresponding points, (*b*) the double points,[2] (*c*) one double point and one corresponding pair.

To find the involution determined by two pairs of

[1] *Cf.* §48, last theorem.

[2] We shall call the fixed points of an involution *double points*. Some writers use the unsatisfactory term *foci*.

corresponding points. If x_1, x_1' and x_2, x_2' are pairs in the involution (1) we must have

$$\begin{aligned} Ax_1x_1' + B(x_1 + x_1') + C &= 0 \\ Ax_2x_2' + B(x_2 + x_2') + C &= 0. \end{aligned}$$

Eliminating the coefficients from these three equations we have the involution in the convenient form

$$\begin{vmatrix} xx' & x + x' & 1 \\ x_1x_1' & x_1 + x_1' & 1 \\ x_2x_2' & x_2 + x_2' & 1 \end{vmatrix} = 0. \tag{2}$$

The condition that a third pair x_3, x_3' be a pair in the involution, *i. e.*, the condition that the three pairs x_1, x_1'; x_2, x_2'; x_3, x_3' belong to an involution is found at once from (2) to be

$$\begin{vmatrix} x_1x_1' & x_1 + x_1' & 1 \\ x_2x_2' & x_2 + x_2' & 1 \\ x_3x_3' & x_3 + x_3' & 1 \end{vmatrix} = 0. \tag{3}$$

Suppose however that the two pairs are not given explicitly but as roots of quadratic equations. For example let us find the involution set up by the pairs of roots of the two equations

$$\begin{aligned} a_1x^2 + b_1x + c_1 &= 0 \\ a_2x^2 + b_2x + c_2 &= 0. \end{aligned} \tag{4}$$

Denoting the roots of these equations respectively by x_1, x_1'; x_2, x_2' we have from algebra

$$\begin{aligned} x_1x_1' &= c_1/a_1, & x_1 + x_1' &= -b_1/a_1 \\ x_2x_2' &= c_2/a_2, & x_2 + x_2' &= -b_2/a_2. \end{aligned}$$

Substituting these in (2) and rearranging we obtain as the required involution

$$\begin{vmatrix} 1 & x + x' & xx' \\ a_1 & -b_1 & c_1 \\ a_2 & -b_2 & c_2 \end{vmatrix} = 0. \tag{5}$$

87. Third definition of involution.—The condition that the three pairs of roots of the quadratics

$$Q_i \equiv a_i x^2 + b_i x + c_i = 0, \qquad i = 1, 2, 3, \tag{1}$$

be in an involution is (§86, equation (5))

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0. \tag{2}$$

But this is just the condition that equations (1) considered as linear in x^2 and x should be dependent or belong to a linear system. In other words we must have a relation

$$k_1 Q_1 + k_2 Q_2 + k_3 Q_3 \equiv 0 \tag{3}$$

where the k's are parameters.

We are thus led to a new definition of an involution. If Q_1 and Q_2 represent two pairs of points, then for every value of k

$$Q_1 + kQ_2 \equiv Q \tag{4}$$

will represent a pair of points in the involution determined by Q_1 and Q_2. For the quadratics Q_1, Q_2, Q satisfy a relation like (3). Or

An involution is a one-parameter family (pencil) of quadratics.

88. A fourth definition of involution is afforded by the following

Theorem.—*All pairs of points x, x' harmonic to two given points a, b are in an involution of which a and b are the double points, and conversely.*

For if $(xx' \mid ab) = -1$ we have

$$(x - a)(x' - b) + (x - b)(x' - a) = 0$$

or

$$2xx' - (a + b)(x + x') + 2ab = 0. \tag{1}$$

From the second equation it is evident that x and x' belong to an involution. While from the first it is plain that a and b are the double points since $x' = a$ when $x = a$ and $x' = b$ when $x = b$.

Moreover since (1) contains two arbitrary constants it represents the general involution, *i. e.*, the converse of the theorem is true.

Since two pairs uniquely determine an involution we may say in virtue of the foregoing theorem: *There is a unique pair of points harmonic to two given pairs, viz., the double points of the involution determined by the two pairs.*

89. If the involution is written in either of the forms (1), (2), (5) §86 the double points are found by setting $x = x'$, whereas in (1) §88 the double points are in evidence.

Let us now consider the converse problem,—*given the double points to find the involution.* If the double points are a and b the involution may be expressed as in the preceding section. But if the double points are given implicitly by the quadratic

$$ax^2 + 2bx + c = 0 \tag{1}$$

how shall we write the involution?

Now

$$axx' + b(x + x') + c = 0 \tag{2}$$

is an involution whose double points are the roots of (1). And since the double points uniquely determine the involution it follows that (2) is none other than the involution in question.

Thus associated with every quadratic equation (1) is a quadratic involution (2). *To derive the involution from the equation giving its double points we have merely to apply the following rule: Replace x^2 by xx' and $2x$ by $x + x'$.*

This process is called *polarizing* the equation, and (2) is the polarized form of (1).

EXERCISES

1. Verify directly that the condition that a collineation (T §85) be of period 2, *i. e.*, that $T^2 = 1$, is $a + d = 0$.

2. Show that a parabolic involution is degenerate. Hence a proper involution has distinct double points.

3. Find the involution determined by the pairs 1,3 $0, -2$; $x^2 - x + 1 = 0$, $x^2 + 3x + 2 = 0$. What are the double points of the involutions?

4. Find the involutions whose double points are $0, \infty$; ± 1, $\pm i$. Show that these three pairs of points are mutually harmonic, *i. e.*, that each pair are the double points of the involution determined by the other two.

5. Find the condition that the quadratics $a_1x^2 + 2b_1x + c_1 = 0$, $a_2x^2 + 2b_2x + c_2 = 0$ represent harmonic pairs of points. What is the condition that the roots of a quadratic be self-harmonic?

6. Show that the condition that the identity $ax^2 + 2bx + c \equiv k_1(x - x_1)^2 + k_2(x - x_2)^2$ may hold, *i. e.*, that a quadratic may be written as a sum of squares of linear expressions, is that x_1, x_2 belong to an involution whose double points are given by the quadratic.

7. Find the pair of points harmonic to the pairs 2,0 $3, -1$; 1,3 0,4; 1,5 2,4; $3x^2 - x - 2 = 0$, $8x^2 + 14x + 3 = 0$; $5x^2 + 2x + 10 = 0$, $3x^2 - 10x - 8 = 0$.

8. Write the involutions whose double points are given by the equations $x^2 + x + 1 = 0$, $2x^2 - 3x = 0$, $2x - 1 = 0$.

9. Pairs of points on a line which are conjugate with respect to a conic belong to an involution whose double points are the intersections of line and conic. Dualize.

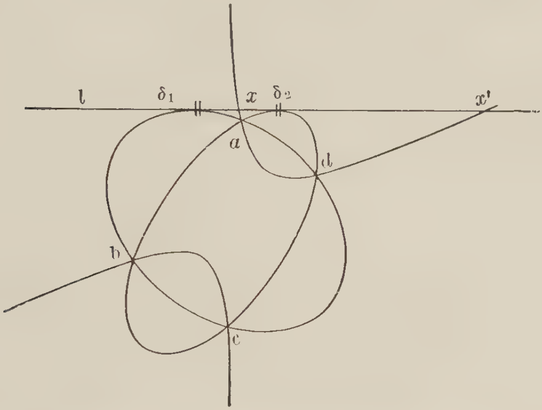
10. Obtain the collineation determined by three pairs of points as a 4-row determinant.

11. Show that the double points of the involution $u + kv$, where $u = a_1x_1^2 + 2b_1x_1x_2 + c_1x_2^2$, $v = a_2x_1^2 + 2b_2x_1x_2 + c_2x_2^2$, are given by the determinant $|uv| \equiv u_1v_2 - u_2v_1 = 0$, where u_1 denotes the partial derivative of u with respect to x_1, etc.

12. If three quadratics have a common factor they belong to a parabolic involution.

13. Find the relation of the two centers of projection (§80) when the collineation is an involution. Verify by repeated constructions that if the collineation is of period two for one point of the line it is so for any point.

90. Consider the system of conics on four points $a\ b\ c\ d$. Any point x of an arbitrary line l determines a unique member of the pencil, namely the conic $abcdx$. But this conic cuts the line in a second point x', *i. e.*, the conic is on $abcdxx'$. On the other hand if we ask for the member of the



pencil which contains x' we obtain the very same conic since it has in common the five points $abcdx'$.

The correspondence between x and x' is both one-to-one and reciprocal for not only does x uniquely determine x' but x' uniquely determines x by the same process. In other words the points are a conjugate pair in an involution. The result may be stated in a second

Theorem of Desargues.—*Conics of a pencil cut an arbitrary line in pairs of points in an involution.*

It is geometrically evident that x and x' will coincide, giving rise to a double point of the involution, when a conic of the pencil is tangent to the line. Since an involution has two double points we have a proof of the theorem (Ex. 2, §15): Among the conics of a pencil two touch an arbitrary line.

Analytic proof of Desargues' theorem. If C and C' are any two conics

$$C \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$
$$C' \equiv a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy$$

then

$$C + kC' = 0 \tag{1}$$

represents the pencil of conics through their common points. Since the equations of the conics are general, the relation of the pencil to a coördinate axis is typical. Setting $z = 0$ we have

$$(ax^2 + 2hxy + by^2) + k(a'x^2 + 2h'xy + b'y^2) \equiv Q + kQ' = 0 \tag{2}$$

to determine the pairs of points in which the pencil cuts the z-axis. Thus the pairs of intersections of the pencil with $z = 0$, *i. e.*, with any line, are given by a pencil of quadratics (2). Hence the pairs belong to an involution. Q. E. D.

91. Special case of Desargues' theorem.—If the four base points of a pencil of conics are considered a quadrangle, the opposite sides of the quadrangle will be degenerate members of the pencil. And the intersection of these sides with any line will belong to the involution set up on the line by the pencil of conics. Hence

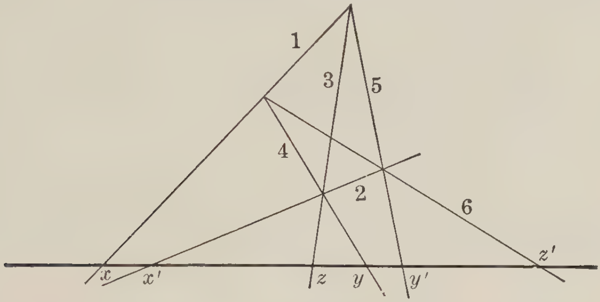
The opposite sides of a complete quadrangle cut an arbitrary line in three pairs of points in an involution.

If the line is on a diagonal point, obviously that point is a double point of the involution. If the line is on two diagonal points, *i. e.*, if it is a diagonal line, we have the familiar harmonic relation of the quadrangle.

We are now able, *given two pairs of points of an involution, to construct the involution.*

Let x, x' and y, y' be pairs of conjugate points of an

involution and let z be any fifth point. To construct the partner z' of z in the involution.



We have but to draw a quadrangle with one pair of opposite sides 1, 2 through x, x', a third side 3 through z and another pair of opposite sides 4, 5 through y, y'. Then the sixth side 6 will cut out the required point z'.

METRICAL PHASES OF COLLINEATIONS AND INVOLUTIONS

92. Metrically canonical forms of collineations.—If x and x' are ordinary Cartesian coördinates on a line, the fixed points of a collineation when reduced to the canonical form

$$x' = kx \tag{1}$$

are the origin and the point at infinity on the line. The collineation now has the effect of multiplying the distance of any point x from the origin by the constant k. Or the result is to magnify the figure of the range in the ratio $k:1$. The collineation is called accordingly a *similitudinous transformation* or a *stretch.*

On the other hand we obtain the canonical form of a parabolic collineation

$$x' = x + k \tag{2}$$

when the fixed points coincide at infinity. This collineation, the result of which is to displace every point x of the line by a constant distance k (in the same direction) is called a *translation.*

93. We note also **some special metrical forms of an involution** obtained by assigning particular values to the coefficients in the general equation

$$Axx' + B(x + x') + C = 0, \tag{1}$$

where of course the variables are to be interpreted as Cartesian coördinates.

1°. Thus

$$x + x' = 0 \tag{2}$$

represents an involution whose double points are the origin and the point at infinity. Conjugate points now lie equally distant from the origin on either side. Hence

All points of a line which lie in pairs at equal distances on opposite sides of a fixed point of the line belong to an involution whose double points are the fixed point and the point at infinity.

2°. Another special form of equation (1) is

$$xx' = k^2 \tag{3}$$

which represents an involution whose double points are $\pm k$. The origin therefore lies midway between the double points and is called in this connection the *center* of the involution.

If

$$xx' = -k^2 \tag{4}$$

the involution is elliptic, the double points being $\pm ik$. The origin however is still called the center. We may summarize then in the theorem

All pairs of collinear points, the product of whose distances from a fixed point is constant, are conjugate pairs of an

involution of which the fixed point is center. The involution is hyperbolic or elliptic according as the constant is positive or negative, i. e., according as conjugate points are on the same or opposite sides of the center.

Elliptic involutions of lines are perhaps best exemplified by the involution consisting of pairs of perpendicular lines on a point. That such lines constitute an elliptic involution follows from the theorem (Ex. 7, §58) that all perpendicular lines are harmonic with the circular points.

Or directly, let the equation of lines through the origin be written in rectangular Cartesian coördinates in the form

$$y = tx \tag{5}$$

where t, the slope, may be regarded as a parameter. Now the slopes t, t' of perpendicular lines are connected by the relation

$$tt' + 1 = 0, \tag{6}$$

a special case of (1). Hence the lines t, t' are conjugate pairs in the involution whose double lines are given by $t^2 + 1 = 0$, or $t = \pm i$. That is

Pairs of perpendicular lines about a point belong to an involution whose double lines are the circular rays meeting at the point.

EXERCISES

1. Dualize Desargues' theorem, the special case and the construction.

2. If a quadrilateral have the circular points as one pair of opposite vertices, the inscribed conics are confocal (Ex. 16, §35). Hence the pairs of tangents from an arbitrary point to a system of confocal conics belong to an involution of lines whose double lines are the tangents to the two conics of the system which are on the point. Show that the double lines of the involution are perpendicular, *i. e.*, that confocal conics intersect orthogonally.

3. Show that the system of confocal conics in Ex. 2 contains no parabolas. By considering the involution determined on the line at

infinity by a pencil of point conics find how many conics of the pencil are parabolas.

4. If a pencil of conics sets up an involution on the line at infinity which has I and J for double points, what can you say of the conics of the pencil?

5. By considering the involution of conjugate diameters of a central conic, show that there is one pair of conjugate diameters which are perpendicular.

6. What is the effect on the involution cut out of a line by a pencil of conics if (a) the line is on a base point of the pencil, (b) two base points?

7. Pairs of points, the ratio of whose distances from a given point is constant, are corresponding pairs in a collineation which has the given point for a fixed point. What is the other fixed point?

8. Pairs of points, the difference of whose distances from a given point is constant, are corresponding pairs in a collineation.

9. A pencil of circles (all circles on two finite points) cut any transversal in pairs of points in an involution whose center is cut out by the common chord of the circles.

10. If a constant angle rotate about the vertex its sides will be corresponding lines in a collineation of lines. What type is the collineation? When will it be an involution, and what then are the double lines?

CHAPTER VIII

BINARY FORMS

94. The analytic geometry of the range, the pencil and certain other one-dimensional manifolds depends on the theory of equations in one variable. But this subject as presented in elementary algebra,—chiefly from the metric point of view,—is inadequate for present purposes since we are here interested in projective results. Accordingly we shall now sketch certain topics in the theory which are indispensable to any projective interpretation.[1]

A rational integral homogeneous algebraic function in any number of variables is called an *algebraic form* or *quantic*.

Forms containing 2, 3, 4, . . . p variables are termed *binary*, *ternary*, *quaternary*, . . . p-ary. And forms of degree 1, 2, 3, 4, . . . n are called *linear*, *quadratic*, *cubic*, *quartic*, . . . n-ic.

In the representation of algebraic forms there is usually a distinct advantage in employing binomial or multinomial coefficients though they may be omitted when convenient. Thus we may write the binary n-ic

$$f \equiv a_0 x_1^n + \binom{n}{1} a_1 x_1^{n-1} x_2 + \binom{n}{2} a_2 x_1^{n-2} x_2^2 + \ . \ . + a_n x_2^n$$

[1] For a comprehensive account of this important branch of mathematics the student should consult the standard treatises: Salmon, *Modern Higher Algebra*, Elliott, *Algebra of Quantics*, Grace and Young, *Algebra of Invariants* in English or Clebsch, *Binären Formen*, Gordan, *Invariantentheorie* and Meyer, *Formen und Invariantentheorie* in German. Shorter accounts will be found in Glenn, *Theory of Invariants* and Dickson, *Algebraic Invariants*. The books by Salmon, Clebsch and Gordan are unfortunately out of print and rare.

where $\binom{n}{r}$ is the coefficient of the $(r + 1)$th term in the expansion of $(x_1 + x_2)^n$.

The student will note the distinction between an algebraic form and an equation in the same number of variables. A form is the function itself whereas the corresponding equation is the form set equal to zero.

The equation $f = 0$ may be said to represent n points of a range or n lines of a pencil, namely the points or lines whose homogeneous coördinates satisfy the equation. We may therefore without danger of confusion refer to these elements concisely as the n points f or the n lines f.

But $f = 0$ may represent elements in other one-dimensional manifolds, *viz.*, those manifolds whose elements can be placed in (1, 1) correspondence with the numbers (ratios x_1/x_2) in the (one-dimensional) number system.[1] The aggregate of these one-dimensional realms whose elements can be regarded as concrete representations of binary forms, —and in the abstract may be considered identical with the number (coördinate) system which they have in common,— constitute the *binary domain*. The complete theory of any one of these realms,—as, *e. g.*, the algebra of binary forms or the geometry of the line,—comprises in the abstract the theory of the whole binary domain.

Two systems like those under consideration which are merely different representations of the same abstract theory are described as *isomorphic*.[2] In this language the binary domain consists of all those one-dimensional manifolds which are isomorphic with the range.

95. Polar forms.—An important class of forms associated

[1] We saw (§64) that the conic has this property. More generally any rational curve,—whether generated by points, lines, planes or spaces of higher dimensions,—of any degree, in the plane or space of n dimensions is such a manifold.

[2] *Cf.* dual geometries under the principle of duality.

with any binary form can be derived as follows. Consider the differential symbol

$$y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} \equiv \left(y \frac{\partial}{\partial x}\right) \tag{1}$$

which is called a *polar operator.* If now we operate with this symbol on f, a binary form of order n, considering the y's independent of the x's thus

$$\left(y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2}\right) f \equiv y_1 \frac{\partial f}{\partial x_1} + y_2 \frac{\partial f}{\partial x_2} \equiv f_y \tag{2}$$

we obviously obtain a binary form f_y of order $n - 1$, which is called the (first) *polar* of (y_1, y_2) with respect to f.[1] The process is called *polarizing* the form f with respect to y.

Geometrically this polar form represents $n - 1$ points which constitute a polar set of the point y with respect to the n points f.

Polarizing f_y with respect to y' we derive another polar form $f_{yy'}$, which is called the *mixed polar* of y and y' with respect to f. Since successive partial differentiation of polynomials is commutative, we have

$$\begin{aligned} f_{y'y} = f_{yy'} &\equiv \left(y_1' \frac{\partial}{\partial x_1} + y_2' \frac{\partial}{\partial x_2}\right)\left(y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2}\right) f \\ &\equiv y_1 y_1' \frac{\partial^2 f}{\partial x_1^2} + (y_1 y_2' + y_2 y_1') \frac{\partial^2 f}{\partial x_1 \partial x_2} + y_2 y_2' \frac{\partial^2 f}{\partial x_2^2}. \end{aligned} \tag{3}$$

Hence alternative methods of calculating the mixed polar are (*a*) polarize f as to y and then polarize the resulting form as to y', or the reverse, (*b*) multiply the differential symbols[2] and apply the resulting operator to f as in the last expression on the right above. The second method is usually simpler in practice.

Speaking geometrically, we may continue the process of

[1] Also called the first polar of f with respect to y_1, y_2.

[2] This multiplication is performed as if the operators were ordinary algebraic expressions but in the product the powers of $\frac{\partial}{\partial x}$, etc., are to be interpreted as indicating successive differentiation.

polarizing f as to the set of points y, y', y'' . . . y^r so long as $r \leqq n$. The mixed polar of r points is a set of $n - r$ points since each polarizing operation reduces the degree of f by one. A case of particular importance is when $r = n$. Then the polar of the n points y will not involve x being a function of the y's and the coefficients of f only, and f is said to be *completely polarized.*

96. Special polar forms.—Instead of taking mixed polars of two or more points we may form repeated polars of the same point. Thus if $y = y'$ in (3) of the preceding section we have

$$\left(y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2}\right)^2 f \equiv y_1^2 \frac{\partial^2 f}{\partial x_1^2} + 2y_1y_2 \frac{\partial^2 f}{\partial x_1 \partial x_2} + y_2^2 \frac{\partial^2 f}{\partial x_2^2} \quad (1)$$

which is called the *second polar* of y with respect to f.

1°. Generally

$$\left(y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2}\right)^r f \quad (2)$$

is called the rth polar of y with respect to f and f is said to be polarized r times with respect to y.[1] Obviously the rth polar represents a set of $n - r$ points, the nth polar being a constant.

A classical example of repeated polars is found in the expansion of a form by Taylor's theorem. For if $f(x_1, x_2) \equiv f$ is a binary form of order n we have by Taylor's formula

$$f(x_1 + y_1, x_2 + y_2) = f(x_1, x_2) + \left(y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2}\right) f$$
$$+ \frac{1}{2!}\left(y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2}\right)^2 f + \ . \ . \ . + \frac{1}{r!}\left(y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2}\right)^r f$$
$$+ \ . \ . \ . + \frac{1}{n!}\left(y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2}\right)^n f, \quad (3)$$

each term of which is a polar of f with respect to y.

[1] It is usual to introduce a numerical factor $\frac{(n-r)!}{n!}$ to neutralize that arising from differentiation, but as we are chiefly concerned with the geometry this is of little consequence.

Now interchanging x and y we obtain the expansion in the equivalent form

$$f(y_1 + x_1, y_2 + x_2) = f(y_1, y_2) + \left(x_1 \frac{\partial}{\partial y_1} + x_2 \frac{\partial}{\partial y_2} \right) f + \cdots \frac{1}{(n-r)!} \left(x_1 \frac{\partial}{\partial y_1} + x_2 \frac{\partial}{\partial y_2} \right)^{n-r} f + \cdots + \frac{1}{n!} \left(x_1 \frac{\partial}{\partial y_1} + x_2 \frac{\partial}{\partial y_2} \right)^{n} f. \quad (4)$$

Since (3) and (4) are identical we have, equating like terms

$$f(x_1, x_2) = \frac{1}{n!} \left(x_1 \frac{\partial}{\partial y_1} + x_2 \frac{\partial}{\partial y_2} \right)^{n} f(y_1, y_2)$$

$$\left(y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} \right) f(x_1, x_2) = \frac{1}{(n-1)!} \left(x_1 \frac{\partial}{\partial y_1} + x_2 \frac{\partial}{\partial y_2} \right)^{n-1} f(y_1, y_2)$$

$$\frac{1}{r!} \left(y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} \right)^{r} f(x_1, x_2) = \frac{1}{(n-r)!} \left(x_1 \frac{\partial}{\partial y_1} + x_2 \frac{\partial}{\partial y_2} \right)^{n-r} f(y_1, y_2)$$

That is, *the rth polar of* (y_1, y_2) *with respect to* $f(x_1, x_2)$ *is equal to the* $(n - r)$*th polar of* (x_1, x_2) *with respect to* $f(y_1, y_2)$, a theorem of great utility in calculation. *Thus to find the* $(n - 2)$*th polar of* (y_1, y_2) *with respect to* $f(x_1, x_2)$ *we calculate the second polar and interchange the* x*'s and* y*'s.*

Geometrically, *if* x *is a point of the rth polar of* y, *then* y *is a point of the* $(n - r)$*th polar of* x.

2°. Again f may be polarized r times with respect to y, r' times with respect to y', etc. where the r's may have any integral values from 1 up to n provided only that $r + r' + \cdots \leqq n$. We thus obtain a mixed polar with some of the points repeated.

3°. Partial derivatives of f with respect to the variables are themselves special cases of polar forms. Thus

$\dfrac{\partial^r f}{\partial x_1{}^r}$ is the rth polar of (1, 0),

$\dfrac{\partial^r f}{\partial x_2{}^r}$ is the rth polar of (0, 1)

and

$\dfrac{\partial^{r+s} f}{\partial x_1{}^r \partial x_2{}^s}$ is a mixed polar, found by polarizing r times with respect to (1, 0) and s times with respect to (0, 1)

97. **Euler's theorem for homogeneous functions.**—For the binary form f the theorem is

$$\left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\right) f = nf.$$

If f is supposed written without binomial coefficients, which in no way affects the proof, the general term may be taken as

$$t_k = a_k x_1{}^{n-k} x_2{}^k.$$

Then

$$\frac{\partial t_k}{\partial x_1} = (n - k) a_k x_1{}^{n-k-1} x_2{}^k,$$

$$\frac{\partial t_k}{\partial x_2} = k a_k x_1{}^{n-k} x_2{}^{k-1},$$

whence

$$\left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\right) t_k = n a_k x_1{}^{n-k} x_2{}^k = n t_k.$$

The result is to multiply each term of f and therefore f itself by n. Q. E. D.

Applying the theorem successively we have

$$\left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}\right)^r f = n(n - 1) \; . \; . \; . \; (n - r + 1) f.$$

This does not mean that the rth polar of a set of n points f with respect to a point of the set is the original set. For

the rth polar here as elsewhere is a set of $n - r$ points. What it does mean is that the rth polar of a set of n points with respect to a point (a, b) of the set will contain (a, b) as one of its $n - r$ points.[1] Or

The locus of a point which is a part of its own rth polar with respect to a set of n points f is f.

It follows immediately that *the nth polar of any one of the points f with respect to f vanishes, and f is the locus of such points.*

There is no difficulty in extending Euler's theorem to algebraic forms in any number of variables.[2]

EXERCISES

1. Polarize completely with respect to y the cubic $f = a_0x_1^3 + 3a_1x_1^2x_2 + 3a_2x_1x_2^2 + a_3x_2^3$.

2. Find the first and second polars of the cubic in Ex. 1 with respect to (y_1, y_2). Show that the first can be obtained from the polarized form of the cubic by setting two y's equal and then replacing them by x; also that the second can be found by replacing one y by x in the polarized form and setting the other y's equal.

3. The polar of a point (y_1, y_2) with respect to a pair of points q is the harmonic conjugate of y with respect to q. And the polars of variable points y of the range with respect to q form with the points themselves conjugate pairs in an involution whose double points are q. *Cf.* §43.

4. Find the polars of (1, 0), (1, 1), (0, 1) with respect to $x_1^3 + 6x_1^2x_2 - 12x_1x_2^2 + 5x_2^3$ and show that they belong to an involution. Find its double points.

5. The polars of the points of the range with respect to the three points of f of Ex. 1 constitute an involution. Find the involution and

[1] The difficulty arises from the fact that x_1, x_2 as coefficients in the polar operator play a different rôle from the x_1, x_2 in f. In the one place they represent a definite one of the points f, in the other they refer to any of the n points. An example will make this clear. The polar of (1, 1) with respect to the three points $x_1^3 - x_2^3$ of which it is obviously one is $3(x_1^2 - x_2^2)$. The polar is thus the point itself and another point.

[2] Indeed it is valid for any homogeneous function, a function $f(x, y, z, \ldots)$ being defined as homogeneous, of degree n if $f(tx, ty, tz \ldots) = t^nf(x, y, z \ldots)$.

its double points. Suggestion: Consider the involution as determined by $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}$.

6. Show that the polar of one root of a binary cubic with respect to the cubic consists of the root itself and the polar of the root with respect to the other two roots.

7. Find the $(n-1)$th and the $(n-2)$th polars of (y_1, y_2) with respect to the binary sextic $a_0x_1^6 + \ldots$

8. Prove Euler's theorem for ternary forms. What is the geometric meaning?

9. Verify directly that the rth polar of $(0, 1)$ with respect to $a_0x_1^n + na_1x_1^{n-1}x_2 + \ldots + na_{n-1}x_1x_2^{n-1}$ contains $(0, 1)$ as a part of it and that the nth polar vanishes.

10. Show that all the second polars of $x_1^4 - x_2^4$ belong to an involution of which the points themselves are two conjugate pairs.

11. Find the condition that the three polars $\frac{\partial^2 f}{\partial x_1^2}, \frac{\partial^2 f}{\partial x_1 \partial x_2}, \frac{\partial^2 f}{\partial x_2^2}$ of the binary quartic $f = a_0x_1^4 + \ldots$ belong to an involution. Show that the quartic in Ex. 10 satisfies this condition.

98. Polar of one binary form with respect to another. We have supposed in the foregoing that the coördinates of the points whose polars were sought were known. Let us see what modification is required when the points are given implicitly as roots of an equation.

To find the polar of a single point (y_1, y_2) where $ay_1 + by_2 = 0$ we first solve the equation, obtaining

$$\frac{y_1}{y_2} = \frac{-b}{a}.$$

Hence the polar operator

$$y_1\frac{\partial}{\partial x_1} + y_2\frac{\partial}{\partial x_2} \text{ becomes } a\frac{\partial}{\partial x_2} - b\frac{\partial}{\partial x_1}.$$

Comparing this with the equation we see that the net result has been to replace y_1 by $\frac{\partial}{\partial x_2}$ and y_2 by $-\frac{\partial}{\partial x_1}$. That is, to find the polar of a point with respect to any set of points f when the coördinates of the point are given implicitly

by a linear form, we substitute for the variables differential symbols as indicated and operate on f.

Consider now a group of m points represented by a form $\phi\ (y_1,\ y_2)$. Let it be required to find the mixed polar of this set of points with respect to a second set of n points $f(x_1,\ x_2)$, $m \leqq n$. The mixed polar is formed by taking the polar of each point y successively with respect to f, *i. e.*, by taking the polar of y with respect to f, then the polar of y' with respect to this polar, etc. until all the points y have been used. Such a polar is described concisely as the *polar of ϕ with respect to f.* We shall show that the artifice employed in the case of a linear form applies in general. For by the fundamental theorem of algebra, the form ϕ can be resolved into m linear factors. Suppose

$$\phi \equiv (a_1y_1 + b_1y_2)(a_2y_1 + b_2y_2)\ \ .\ \ .\ \ .\ (a_my_1 + b_my_2). \quad (1)$$

Evidently the polar of ϕ is found by substituting for y_1, y_2 in (1) differential symbols as above and operating on f. But the result of the substitution is the same whether ϕ is actually expressed as a product of factors or not. Hence we have the general rule: *To find the polar of $\phi(y_1,\ y_2)$ with respect to $f(x_1,\ x_2)$, replace y_1 by $\dfrac{\partial}{\partial x_2}$ and y_2 by $-\dfrac{\partial}{\partial x_1}$ in ϕ and operate on f.* For convenience however, we shall usually write $\dfrac{\partial}{\partial x_1} = \xi_1$ and $\dfrac{\partial}{\partial x_2} = \xi_2$.

Thus the polar of a cubic

$$\phi \equiv b_0y_1^3 + 3b_1y_1^2y_2 + 3b_2y_1y_2^2 + b_3y_2^3$$

with respect to a second cubic

$$f \equiv a_0x_1^3 + 3a_1x_1^2x_2 + 3a_2x_1x_2^2 + a_3x_2^3$$

is

$$(-b_3\xi_1^3 + 3b_2\xi_1^2\xi_2 - 3b_1\xi_1\xi_2^2 + b_0\xi_2^3)f = 6(-b_3a_0 + 3b_2a_1 - 3b_1a_2 + b_0a_3). \quad (2)$$

Suppose now that f and ϕ are made non-homogeneous by dropping the subscript 1 and setting x_2 and y_2 equal to 1. Denote the roots of the corresponding equations by x_1, x_2, x_3 and y_1, y_2, y_3 and let s_i and σ_i ($i = 1, 2, 3$) represent the ordinary symmetric functions of these roots thus[1]

$$\begin{aligned} s_1 &= x_1 + x_2 + x_3 & \sigma_1 &= y_1 + y_2 + y_3 \\ s_2 &= x_2x_3 + x_3x_1 + x_1x_2 & \sigma_2 &= y_2y_3 + y_3y_1 + y_1y_2 \\ s_3 &= x_1x_2x_3 & \sigma_3 &= y_1y_2y_3. \end{aligned}$$

Then the right side of (2), neglecting a numerical factor may be written in either of the forms

$$\begin{gathered} a_0\sigma_3 + a_1\sigma_2 + a_2\sigma_1 + a_3 \\ b_0s_3 + b_1s_2 + b_2s_1 + b_3 \\ \sigma_3 - \frac{\sigma_2 s_1}{3} + \frac{\sigma_1 s_2}{3} - s_3. \end{gathered} \tag{3}$$

The first of these forms suggests a simple *rule for finding the polar of a binary form ϕ with respect to another form f of the same order: Replace x^i in f by $\dfrac{\sigma_i}{\binom{n}{i}}$ from ϕ.*

From the symmetry of (2) and the last expression in (3) it is obvious that the polar relation of forms of the same order is reciprocal, *i. e.*, the polar of ϕ with respect to f is identical with the polar of f with respect to ϕ.

When f and ϕ are the same form the method explained gives the *polar of a form with respect to itself.*

99. Apolar forms.—When the polar of one form with respect to another vanishes identically, *i. e.*, when every

[1] It is not necessary to discard the homogeneous form. Thus if the values of the ratio x_1/x_2 in $f = 0$ are z_1/z_2, z_1'/z_2', z_1''/z_2'' and we write $s_1 = z_1z_2'z_2'' + z_1'z_2z_2'' + z_1''z_2z_2'$, $s_2 = z_1'z_1''z_2 + z_1''z_1z_2' + z_1z_1'z_2''$, $s_3 = z_1z_1'z_1''$ (together with similar expressions for the σ's) equations (3) and the rule following are still valid. Likewise equation (2) §99 with appropriate interpretations for the s's and the σ's holds either for homogeneous or non-homogeneous forms. It is usually immaterial which form is employed unless the symmetric functions are to be written at full length when of course the non-homogeneous forms are much simpler.

coefficient is zero, the two forms are said to be *apolar*.[1] Now the polar of ϕ^m with respect to f^n, $m < n$, is a form of order $n - m$ containing therefore $n - m + 1$ coefficients. Consequently it is in general precisely $n - m + 1$ conditions on the two to be apolar. In particular if the forms are of the same order it is one condition for them to be apolar. These conditions may be imposed on either of the two forms or upon both jointly. The equations of condition are linear in the coefficients of each form, a fact of importance in the applications.

Geometrically we speak of the points represented by apolar forms f and ϕ as *apolar sets of points*. When $m<n$ it is plain that the m points ϕ and an *arbitrary* set of $n - m$ points constitute a set apolar to the n points f.

The condition that two forms of the same order be apolar can be written down at once by generalizing (2) or (3) of the preceding section. If the forms are

$$\begin{aligned} f &\equiv a_0x_1^n + na_1x_1^{n-1}x_2 + \ . \ . \ . \ + a_nx_2^n \\ \phi &\equiv b_0x_1^n + nb_1x_1^{n-1}x_2 + \ . \ . \ . \ + b_nx_2^n \end{aligned}$$

the apolarity condition is

$$a_0b_n - na_1b_{n-1} + \binom{n}{2} a_2b_{n-2} + \ . \ . \ . \ (-)^n a_nb_0 = 0 \quad (1)$$

or

$$\sigma_n - \frac{\sigma_{n-1}s_1}{n} + \ . \ . \ . \ (-)^i \frac{\sigma_{n-i}s_i}{\binom{n}{i}} (-)^{i+1} \ . \ . \ . \ (-)^n s_n = 0, \quad (2)$$

where s and σ refer to the symmetric functions of the roots of $f = 0$ and $\phi = 0$.

If f and ϕ are the same form so that $a_i = b_i$ and n is odd it is readily seen that the terms of (1) pair off and annul each other. Hence

1°. *A form of odd order is always apolar to itself.*

[1] Greek α privative, *without* + *polar*.

But it is a condition for a form of even order to be self-apolar, since (1) will then contain an odd number of terms which double up, like terms having the same sign, except of course the middle term. The condition that an even form f of order $2p$ be self-apolar is found therefore by setting $a_i = b_i$ in (1) and writing down the first p terms just as they occur and adding *half* the middle term.

2°. *If ϕ is apolar to several forms $f_0, f_1, \ldots f_r$, it is apolar to any linear system of the f's*

$$f_0 + k_1 f_1 + \ldots + k_r f_r.$$

For by substituting the usual differential symbols (§98) in ϕ and operating on the linear system every term vanishes by hypothesis.

3°. *There is a unique form of order n apolar to n linearly independent forms of order n, and conversely.*

For it is n linear conditions on a form ϕ^n to be apolar to the n forms $f_1, f_2, \ldots f_n$ of order n. And these conditions are linearly independent if the f's are,[1] hence they just suffice to determine the n essential constants in ϕ.

To prove the converse of the theorem observe that there cannot be more than $n + 1$ linearly independent forms of a given order since any form of order n can be written as a linear combination of the $n + 1$ linearly independent forms

[1] It will be sufficient to prove this for the case of the three cubics

$$f_i \equiv a_i x_1^3 + b_i x_1^2 x_2 + c_i x_1 x_2^2 + d_i x_2^3, \qquad i = 1, 2, 3.$$

The condition that

$$\phi \equiv \alpha x_1^3 + 3\beta x_1^2 x_2 + 3\gamma x_1 x_2^2 + \delta x_2^3$$

be apolar to each of the f's is

$$q_i \equiv a_i \delta - b_i \gamma + c_i \beta - d_i \alpha = 0.$$

If these equations are not linearly independent, three numbers k_i can be found such that

$$(kq) \equiv (ka)\delta - (kb)\gamma + (kc)\beta - (kd)\alpha \equiv 0,$$

where the parentheses denote row products, thus $(ka) \equiv k_1 a_1 + k_2 a_2 + k_3 a_3$.

But this implies that

$$(kf) \equiv (ka)x_1^3 + (kb)x_1^2 x_2 + (kc)x_1 x_2^2 + (kd)x_2^3 \equiv 0,$$

i. e., that the f's are not linearly independent.

$x_1^{n-r}x_2^r$, $r = 0, 1, \ldots n$. Hence all forms of order n belong to the n-parameter family

$$k_0f_0 + k_1f_1 + k_2f_2 + \ldots + k_nf_n$$

built on the $n + 1$ linearly independent base forms f. To ask that this system be apolar to ϕ is to impose a linear condition on the k's which enables us to eliminate one of them as k_0. Hence all forms apolar to ϕ belong to an $(n - 1)$-parameter family

$$k_1f_1' + k_2f_2' + \ldots + k_nf_n'$$

built on the n linearly independent base forms f', *each of which is apolar to* ϕ.

This not only proves the theorem but actually supplies a system of n linearly independent apolar forms, namely the forms f'.

Thus to find three linearly independent cubics apolar to the cubic

$$x_1^3 + 3x_1x_2^2$$

we write down the general system of cubics

$$k_0x_1^3 + k_1x_1^2x_2 + k_2x_1x_2^2 + k_3x_2^3.$$

The condition that the given cubic be apolar to this system is $k_1 + k_3 = 0$ and the system reduces to

$$k_0x_1^3 + k_1(x_1^2x_2 - x_2^3) + k_2x_1x_2^2$$

whose base forms constitute a set of the kind required.[1]

The last theorem of §97 may be stated in the language of the present section as follows.

4°. *The nth power of any linear factor of f is apolar to f and the linear factors of f comprise all such linear forms.* Or geometrically, *the locus of points which taken n times are apolar to a set of n points f consists of the points f themselves.*

[1] It is important to note that this method is available whether ϕ has repeated factors or not. Grace and Young, §185, *e. g.* require different methods according as the factors of ϕ are distinct or not.

If f contain no repeated factors the nth powers of its linear factors are a set of n linearly independent forms apolar to f.

5°. *If ϕ^m is apolar to f^n, $m < n$, ϕ is apolar to any polar (mixed or repeated) of f of order m, and conversely.*

Let ψ^{n-m} be an arbitrary set of $n - m$ points and denote the polar of ψ as to f by p^m. We are to show that ϕ and p are apolar.

By hypothesis (a) the polar of ϕ (as to f) $\equiv 0$, *i. e.*, (b) the mixed polar of ϕ and ψ (as to f) $\equiv 0$. But

$$\text{polar of } \phi \text{ and } \psi \text{ as to } f \equiv \text{polar of } \phi \text{ as to } p.$$

Since the left side vanishes identically (by (b)) the right side also vanishes identically which proves the theorem.

The converse is left as an exercise.

100. Applications to the binary cubic and quartic.—*There is a unique quadratic apolar to a given cubic.* For if the cubic is supposed known there are two linear conditions on the coefficients of the quadratic which determine them uniquely.

To find the apolar quadratic of a cubic, let the cubic be

$$f \equiv a_0x_1^3 + 3a_1x_1^2x_2 + 3a_2x_1x_2^2 + a_3x_2^3 \tag{1}$$

where the a's are fixed. And take for the quadratic

$$\phi \equiv \alpha x_1^2 + \beta x_1x_2 + \gamma x_2^2 \tag{2}$$

where the coefficients are to be determined. Substituting differential symbols in ϕ and operating on f we have as the polar of ϕ with respect to f

$$(\alpha\xi_2^2 - \beta\xi_2\xi_1 + \gamma\xi_1^2)f \equiv 6\{(a_0\gamma - a_1\beta + a_2\alpha)x_1 + (a_1\gamma - a_2\beta + a_3\alpha)x_2\}. \tag{3}$$

The conditions for apolarity are the vanishing of both coefficients in (3) or

$$\begin{aligned} a_0\gamma - a_1\beta + a_2\alpha &= 0 \\ a_1\gamma - a_2\beta + a_3\alpha &= 0. \end{aligned} \tag{4}$$

Eliminating γ, β, α from these equations and $\phi = 0$ and

changing the sign we obtain the required quadratic in the form

$$\begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ x_2^2 & -x_1x_2 & x_1^2 \end{vmatrix}. \tag{5}$$

The apolar quadratic which is of fundamental importance in the projective theory of the binary cubic is called the *Hessian*[1] or the *canonizant* of the cubic.

While a given cubic has a unique Hessian, a specified quadratic is Hessian to a single infinity of cubics. For if a fixed quadratic ϕ be apolar to a general cubic f the coefficients of f are subjected to only two conditions (4) which determine but two of the three essential constants.

It is three conditions for a quadratic to be apolar to a quartic. Now only two of these can be charged to the quadratic so there can be no apolar quadratic unless the quartic itself satisfy a condition. To find this condition suppose the quartic is

$$f \equiv a_0x_1^4 + 4a_1x_1^3x_2 + 6a_2x_1^2x_2^2 + 4a_3x_1x_2^3 + a_4x_2^4. \tag{6}$$

Operating on this with ϕ we have

$$(\gamma\xi_1^2 - \beta\xi_1\xi_2 + \alpha\xi_2^2)f \equiv 12\{(a_0\gamma - a_1\beta + a_2\alpha)x_1^2 + (a_1\gamma - a_2\beta + a_3\alpha)x_1x_2 + (a_2\gamma - a_3\beta + a_4\alpha)x_2^2\}. \tag{7}$$

This will vanish identically if

$$\begin{aligned} a_0\gamma - a_1\beta + a_2\alpha &= 0 \\ a_1\gamma - a_2\beta + a_3\alpha &= 0 \\ a_2\gamma - a_3\beta + a_4\alpha &= 0. \end{aligned} \tag{8}$$

Eliminating γ, β, α we have as the necessary condition that the quartic (6) have an apolar quadratic

$$\begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} = 0,$$

a symmetrical determinant easily remembered.

[1] After the German geometer Hesse.

Ex.—Prove the condition sufficient.

This determinant, the vanishing of which expresses the condition that the quartic have an apolar quadratic, is called the *catalecticant* of the quartic and a quartic for which it vanishes is said to be *catalectic.*

The results obtained in this section for the binary cubic and quartic are characteristic respectively of forms of odd and even order. Thus we have by immediate extension

A form f^{2n-1} *has a unique apolar* ϕ^n, *the canonizant. While a form* f^{2n} *cannot have an apolar* ϕ^n *unless a certain determinant, the catalecticant, vanishes.*

EXERCISES

1. Write the apolarity conditions (1) and (2) §99 when one only of the forms has binomial coefficients, when neither has. Write the apolarity conditions for forms of the nth order from the first two expressions of (3) §98.

2. Find the polar of $a_1x^2 + 2b_1x + c_1$ with respect to $a_2x^2 + 2b_2x + c_2$. Show that two apolar quadratics represent harmonic pairs of points.

3. Prove the converse of 2°, §99.

4. Show that if $m < n$ there are $n - m + 1$ linearly independent forms of order n apolar to a given form of order m.

5. Find a set of six linearly independent forms apolar to the sextic $x_1^3(x_1 + x_2)^2x_2$.

6. Find the cubic apolar to the three cubics, $3t^2$, $3t$, $t^3 + 1$.

7. Find the condition that a quadratic be self-apolar, a quartic, a sextic.

8. Show that the Hessian of a cubic is apolar to all first polars. Connect this with Ex. 5, §97.

9. Find the pencil of cubics of which x_1x_2 is the Hessian.

10. Find the canonizant of the general quintic, of the quintic $k_0t^5 + k_1(t - 1)^5 + k_2$. Write the catalecticant of the sextic.

11. Show that the condition that the second polars of a quartic belong to an involution is that the catalecticant vanish. (*Cf.* Ex. 11, §97.)

12. There is a pencil of quartics apolar to three linearly independent quartics. Find the pencil for the following sets of quartics: (a) $x_1^3x_2$,

$x_1x_2^3$, $x_1^4 + ax_1^2x_2^2 + x_2^4$, (*b*) $ax_1^3x_2 + x_2^4$, $x_1^4 + ax_1x_2^3$, $x_1^2x_2^2$, (*c*) $ax_1^4 + 4bx_1^3x_2$, $4bx_1^3x_2 + 6cx_1^2x_2^2 + 4dx_1x_2^3$, $4dx_1x_2^3 + ex_2^4$.

13. Show that a pencil of quartics contains two self-apolar and three catalectic quartics.

14. Prove that if $a\ b\ c\ d\ x$ are apolar to $a\ b\ c\ d\ y$, either $x = y$ or $a\ b\ c\ d$ form a self-apolar set.

15. If f_1, f_2, f_3 are three linearly independent quintics each of which is apolar to the three ϕ_1, ϕ_2, ϕ_3 show that there is a unique quintic common to the systems $\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3$, and $k_1\phi_1 + k_2\phi_2 + k_3\phi_3$.

16. If the three cubics $f_i \equiv a_ix_1^3 + b_ix_1^2x_2 + c_ix_1x_2^2 + d_ix_2^3$, $i = 1, 2, 3$, are connected by an identical relation of the form $(kf) \equiv k_1f_1 + k_2f_2 + k_3f_3 \equiv 0$, then every three row determinant of the matrix

$$\left\|\begin{matrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{matrix}\right\|$$

vanishes and conversely. Generalize.

17. The completely polarized form as to x_1, x_2, x_3, x_4 of the non-homogeneous quartic $f \equiv a_0x^4 + 4a_1x^3 + \ldots$ is (1) $a_0s_4 + a_1s_3 + a_2s_2 + a_3s_1 + a_4$, where the s's are symmetric functions of the x's. Verify the following practical device for obtaining polar forms: To find with respect to f (*a*) the polar of x_1 put $x_2 = x_3 = x_4 = x$, (*b*) the polar of x_1 and x_2, put $x_3 = x_4 = x$, (*c*) the polar of x_1, x_2, x_3, put $x_4 = x$. And to find the second, third and fourth polars of x_1 respectively, set $x_1 = x_2$ in (*b*), $x_1 = x_2 = x_3$ in (*c*) and $x_1 = x_2 = x_3 = x_4$ in (1). Generalize.

18. If s_i refer to symmetric functions of $x_1, x_2, \ldots x_n$ and σ_i to symmetric functions of $x_1, x_2, \ldots x_{n-1}$, then $s_1 = \sigma_1 + x_n$, $s_2 = \sigma_2 + \sigma_1x_n$, $s_3 = \sigma_3 + \sigma_2x_n \ldots s_n = \sigma_{n-1}x_n$. Hence the mixed polar of $x_1, x_2, \ldots x_{n-1}$ with respect to a form f^n of order n is found by replacing the s's by the σ's in the completely polarized form of f and setting $x_n = x$.

19. Using the method of Ex. 17 or Ex. 18, find (*a*) the second polar of x_1, (*b*) the mixed polar of x_1, x_2, x_3, x_4, (*c*) the fourth polar of x_1 as to the quintic $ax^5 + 5bx^4 + 10cx^3 + 10dx^2 + 5ex + f$.

20. If s refer to symmetric functions of $x_1, x_2, \ldots x_n$ and σ to symmetric functions of $x_1, x_2, \ldots x_{n-2}$, write a table of equivalents of the s's in terms of the σ's and x_n, x_{n-1}. Thence find the mixed polar of x_1, x_2, x_3, x_4 with respect to the sextic $ax^6 + 6bx^5 + 15cx^4 + 20dx^3 + 15ex^2 + 6fx + g$; the fourth polar of x_4.

21. Denoting the polar of ϕ as to f by ϕ on f, show that ϕ on $f^n = nf^{n-1}(\phi$ on $f)$ when ϕ is linear. If f is quadratic, f on $f^n = \triangle n^2 f^{n-1}$.

101. Hessians.—We have already generalized for odd forms the apolar quadratic of the cubic,[1] calling the function from that point of view the canonizant. We shall now indicate a second generalization for forms of any order and in any number of variables.

It can be readily verified directly that the apolar quadratic is, neglecting a numerical factor, the determinant

$$\begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}$$

where the subscripts denote partial differentiation. Generally this determinant written for any binary form f is called the *Hessian determinant* or the *Hessian* of f.[2]

Likewise for a ternary form the Hessian is

$$\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$$

and so on for forms in more than three variables. The Hessian of a binary n-ic is thus of order $2(n-2)$ representing a set of $2(n-2)$ points, while that of a ternary n-ic is of order $3(n-2)$ and represents a curve, etc.

To interpret the Hessian geometrically, consider the $(n-2)$th polar of y with respect to a binary form f, recalling (§96)

$$\left(x\frac{\partial}{\partial y}\right)^{n-2} f(y) \equiv \left(y\frac{\partial}{\partial x}\right)^{2} f(x) \equiv$$
$$y_1^2\frac{\partial^2 f}{\partial x_1^2} + 2y_1y_2\frac{\partial^2 f}{\partial x_1 \partial x_2} + y^2\frac{\partial^2 f}{\partial x_2^2}.$$

[1] Last theorem, §100.

[2] The coincidence in the case of the binary cubic of two associated forms distinct in general is not an isolated occurrence. Thus for the quadratic $ax_1^2 + 2bx_1x_2 + cx_2^2$ the same function $ac - b^2$ represents at once the discriminant, the condition for self-apolarity and the Hessian.

The discriminant of this regarded as a quadratic in y is

$$H: \frac{\partial^2 f}{\partial x_1{}^2}\,\frac{\partial^2 f}{\partial x_2{}^2} - \left(\frac{\partial^2 f}{\partial x_1 \partial x_2}\right)^2 = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}$$

the Hessian of f. We have thus a property characteristic of Hessians which we shall state generally:

The Hessian of an algebraic form is the locus of points whose polar quadratics have double points.

102. Jacobians.—Another determinant of great importance is that studied by Jacobi under the name of *functional determinant* but since named in his honor Jacobi's determinant or the *Jacobian.* If u and v are binary forms (not necessarily of the same order) their Jacobian is defined to be

$$\begin{vmatrix} \dfrac{\partial u}{\partial x_1} & \dfrac{\partial v}{\partial x_1} \\ \dfrac{\partial u}{\partial x_2} & \dfrac{\partial v}{\partial x_2} \end{vmatrix} \equiv \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} \equiv \frac{\partial(u, v)}{\partial(x_1, x_2)} \equiv J(u, v).$$

Some properties of the Jacobian.

1°. *If two binary forms have a common factor, their Jacobian has the same factor.*

If u and v are two forms of orders n and m respectively we have from Euler's theorem (§97)

$$\begin{aligned} x_1 u_1 + x_2 u_2 &= nu \\ x_1 v_1 + x_2 v_2 &= mv. \end{aligned} \tag{1}$$

Solving these equations for x_1 and x_2,

$$x_1 J = n v_2 u - m u_2 v \tag{2}$$

$$-x_2 J = n v_1 u - m u_1 v. \tag{3}$$

It follows at once that any values that make both $u = 0$ and $v = 0$ will also make $J = 0$.

2°. *If two binary forms of the same order have a common factor, their Jacobian will contain the same factor repeated.*

If $n = m$, then differentiating partially (2) with respect to x_2 and (3) with respect to x_1 we obtain

$$\begin{aligned} x_1 J_2 &= n(v_{22}u - u_{22}v) \\ -x_2 J_1 &= n(v_{11}u - u_{11}v). \end{aligned} \tag{4}$$

It appears therefore that any value of x_1/x_2 which causes u and v to vanish simultaneously will also cause J_1 and J_2 to vanish. But the condition that $J_1 = 0$ and $J_2 = 0$ have a common root is just the condition that $J = 0$ have a double root[1] or that J have a square factor.

Cor. *The discriminant of the Jacobian of two binary forms of the same order contains the eliminant of the forms as a factor*, since the former vanishes whenever the latter does.

Consider next the pencil of binary n-ics

$$u + kv \tag{5}$$

which we shall call an *involution*[2] and which represents geometrically ∞^1 sets of n points. Among these sets of points there are in general $2(n - 1)$ which have a *double point*[3] since the discriminant of (5) involves k to the degree $2(n - 1)$. Now any repeated root of (5) $= 0$ will be a simple root of the equations

$$u_1 + kv_1 = 0, \qquad u_2 + kv_2 = 0 \tag{6}$$

obtained by partial differentiation. But if these equations are to hold simultaneously we must have, eliminating k

$$J(u, v) \equiv u_1 v_2 - u_2 v_1 = 0$$

an equation of order $2(n - 1)$. Therefore

[1] See any treatise on the theory of equations. The necessary and sufficient condition that an equation $f(x_1, x_2) = 0$ have a double root x_1/x_2, called the *discriminant* of the equation, is the necessary and sufficient condition that the derived equations $\partial f/\partial x_1 = 0$ and $\partial f/\partial x_2 = 0$ have a common root, called the *eliminant* or *resultant* of the derived equations. *Cf.* footnote p. 27.

[2] Since it is an immediate generalization of the pencil of quadratics, §87.

[3] That is, a point formed by the coincidence of two points of a set.

3°. *The Jacobian of two binary forms of the same order represents the double points of the involution determined by the forms.*

The definition and properties of the Jacobian as presented in this section for binary forms can be extended at once to algebraic forms of any order. Thus the Jacobian of three ternary forms u, v, w is

$$\begin{vmatrix} \dfrac{\partial u}{\partial x_1} & \dfrac{\partial v}{\partial x_1} & \dfrac{\partial w}{\partial x_1} \\ \dfrac{\partial u}{\partial x_2} & \dfrac{\partial v}{\partial x_2} & \dfrac{\partial w}{\partial x_2} \\ \dfrac{\partial u}{\partial x_3} & \dfrac{\partial v}{\partial x_3} & \dfrac{\partial w}{\partial x_3} \end{vmatrix} \equiv \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} \equiv \frac{\partial(u, v, w)}{\partial(x_1, x_2, x_3)} \equiv J(u, v, w),$$

and theorem 3° becomes

If $u = 0$, $v = 0$, $w = 0$ are three algebraic curves of the same order, then the Jacobian curve $J(u, v, w) = 0$ is the locus of double points of the net $u + kv + lw = 0$.[1]

EXERCISES

1. The Hessian of an algebraic form $f(x_1, x_2, x_3, \ldots)$ is the Jacobian of the polars $\partial f/\partial x_1$, $\partial f/\partial x_2$, $\partial f/\partial x_3 \ldots$ Hence the Hessian of a binary form represents the double points of the involution $\dfrac{\partial f}{\partial x_1} + \lambda \dfrac{\partial f}{\partial x_2}$.

2. The discriminant of the Hessian of a binary form f contains the discriminant of f as a factor.

3. The discriminant of a binary cubic and its Hessian are the same function. (Compare the degree of each and apply 2.) Hence find the discriminant of the cubic $ax^3 + 3bx^2 + 3cx + d$; of the cubic $ax^3 + bx^2 + cx + d$.

[1] If (y_1, y_2, y_3) is a point on a curve $f(x_1, x_2, x_3) = 0$, then $\partial f/\partial y_1 = 0$, $\partial f/\partial y_2 = 0$, $\partial f/\partial y_3 = 0$ are necessary and sufficient conditions that y be a double point. Thus differentiating the equation of the net as to x_1, x_2, x_3 and eliminating k and l we obtain $J = 0$ as the locus of double points.

4. The resultant of two quadratics is the discriminant of their Jacobian. Thus find the resultant of $a_1x_1^2 + 2b_1x_1x_2 + c_1x_2^2$ and $a_2x_1^2 + 2b_2x_1x_2 + c_2x_2^2$.

5. Find the Hessian of (*a*) the binary quartic $x^4 + 6cx^2y^2 + y^4$, (*b*) of the binary quintic $x^5 + 5bx^4y + 5cxy^4 + y^5$, (*c*) the ternary quadratic $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$, (*d*) the ternary cubic $x^3 + y^3 - 3axyz$, (*e*) the ternary cubic $x^3 + y^3 + z^3$, (*f*) the ternary cubic $x^3 + y^3 + z^3 + 6axyz$.

6. Calculate the following Jacobians: (*a*) $J(x^3 + y^3, xy)$, (*b*) $J(x^3 + y^3, x^3 - y^3)$, (*c*) $J(x^4 + 6cx^2y^2 + y^4$ and its Hessian), (*c*) $J(a_1x^2 + b_1y^2 + c_1z^2, a_2x^2 + b_2y^2 + c_2z^2, a_3x^2 + b_3y^2 + c_3z^2)$.

Prove the following geometric generalizations of 1° and 2°, §102.

7. If $u = 0$, $v = 0$, $w = 0$ are three curves with a common point, then $J(u, v, w) = 0$ contains the same point.

8. If three curves of the same order meet in a common point, the point is a double point on the Jacobian. (Write the equation corresponding to (2) §102, differentiate partially with respect to each of the three variables and from the three resulting equations show that any values that cause u, v, w to vanish will satisfy $J = 0$ and $\partial J/\partial x = 0$, $\partial J/\partial y = 0$, $\partial J/\partial z = 0$.)

9. Show that the Jacobian of three circles is a circle (orthogonal to the three) and the line at infinity. State a corresponding projective theorem.

CHAPTER IX

ALGEBRAIC INVARIANTS

103. Definition of invariants.—We have already had occasion in the chapters on coördinates and collineations to discuss linear transformations of the variables. We shall now consider the effect of such transformations on binary forms, writing the equations homogeneously thus

$$T: \quad \begin{aligned} x_1 &= \alpha X_1 + \beta X_2 \\ x_2 &= \gamma X_1 + \delta X_2 \end{aligned} \tag{1}$$

where the variables are projective coördinates in the binary domain,—the small letters representing the old and the large letters the transformed variables,—and the coefficients are arbitrary constants. The determinant $D \equiv \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}$ is called the determinant or *modulus* of the transformation.

Suppose now that a binary form

$$f(x_1, x_2) \equiv a_0x_1{}^n + a_1x_1{}^{n-1}x_2 + \ . \ . \ . \ + a_nx_2{}^n$$

is transformed by (1) into a second form

$$F(X_1, X_2) \equiv A_0X_1^n + A_1X_1^{n-1}X_2 + \ . \ . \ . \ + A_nX_2^n.$$

Obviously there will be little resemblance between the two forms. For example the individual coefficients of F will differ from the corresponding coefficients in f and the roots of $F = 0$ will be different from the roots of $f = 0$. The question arises whether there might not be combinations of the letters in f which are unaltered by the transformation. The answer is affirmative as we shall presently prove. In the meantime we shall require some definitions.

Any function I of the coefficients of f which is unchanged i. e., invariant under the transformation T,—except indeed by a constant factor depending on the transformation itself,—is called an invariant of f. The factor in every case will be a power of the modulus. Symbolically

$$I(A_0, A_1, \ldots A_n) = D^w I(a_0, a_1, \ldots a_n).$$

If $w = 0$, I is called an *absolute invariant* of f since in that case it is absolutely unaltered by the transformation. The *degree* of the invariant is the degree in the coefficients, and the *weight* is the power to which the modulus enters.

Again if two or more binary forms $f_1 = a_0x_1{}^m + \ldots$ $f_2 = b_0x_1{}^n + \ldots, \ldots f_k = k_0x_1{}^p + \ldots$ are transformed by T into $F_1 = A_0X_1^m + \ldots, F_2 = B_0X_1^n + \ldots, \ldots F_k = K_0X_1^p + \ldots$ then any function I of the coefficients satisfying the equation

$$I(A_0, \ldots A_m, B_0, \ldots B_n, \ldots K_0, \ldots K_p) = D^w I(a_0 \ldots a_m, b_0 \ldots b_n, \ldots k_0 \ldots k_p)$$

is a *joint* or *simultaneous invariant* of the several forms.

Similarly *any function C of the coefficients and variables of f which is invariant, except for a power of the modulus, under a linear transformation T is designated a covariant of f.* Thus $C(A_0, A_1, \ldots A_n, X_1, X_2) = D^r C(a_0, a_1, \ldots a_n, x_1, x_2)$.

As in the case of invariants we may have *absolute covariants* and *simultaneous covariants.*

The degree of a covariant in the coefficients is called its *degree* and the degree in the variables its *order*. A covariant of degree i and order j is frequently denoted by $C_{i,j}$.

Invariants are thus covariants of order zero. On the other hand when the distinction is not to be emphasized invariant is used as a general term to include all invariant functions.[1] In particular the invariants considered here are called algebraic invariants as distinguished for instance

[1] *Concomitant* is also used.

from the modular invariants in arithmetic and differential invariants in the theory of differential equations.

104. The geometrical importance of invariants rests on the fact that the linear transformaton T of the previous section may represent alike a transformation of coördinates or a projection. The vanishing of an invariant therefore implies a restriction of a form f which is (a) independent of the coördinate system in which f is represented and (b) one which persists after projection. In other words *the vanishing of an invariant implies a projective specialization of f.* Likewise *a covariant of f represents a set of points projectively related to the points f.* And so for simultaneous invariants and covariants.

The existence of invariants and covariants is now evident, indeed we are already familiar with certain types of such functions. Thus *the discriminant of a binary form is an invariant.* For the vanishing of the discriminant means that two or more of the points in question coincide, a property manifestly independent of the coördinate system.

Again *the eliminant of a system of forms is a simultaneous invariant of the forms.* For if the eliminant vanishes the forms have a common point, a relation in no way connected with the coördinate system.

As immediate consequences of the definitions we may mention

(a) *Any form is an absolute covariant of itself.*

(b) *Any invariant or covariant of a covariant is a concomitant of the original form.*

The Jacobian of two quadratics is a simultaneous covariant of the forms. For the Jacobian represents a pair of points harmonic to both the original pairs (3°, §102) and the harmonic property is projective.

105. The last example, §104, is a special case of the following more general theorem:

The Jacobian of any two binary forms is a simultaneous covariant of the forms.

We shall prove this directly by showing that if the two forms f and ϕ are transformed into F and Φ by

$$\begin{aligned} x_1 &= \alpha X_1 + \beta X_2 \\ x_2 &= \gamma X_1 + \delta X_2 \end{aligned} \tag{1}$$

then

$$J(F, \Phi) = DJ(f, \phi).$$

Now

$$\frac{\partial F}{\partial X_1} = \frac{\partial F}{\partial x_1}\frac{\partial x_1}{\partial X_1} + \frac{\partial F}{\partial x_2}\frac{\partial x_2}{\partial X_1} = \alpha\frac{\partial F}{\partial x_1} + \gamma\frac{\partial F}{\partial x_2} \text{ (by (1))} = \alpha\frac{\partial f}{\partial x_1} + \gamma\frac{\partial f}{\partial x_2}. \tag{2}$$

Similarly

$$\frac{\partial F}{\partial X_2} = \beta\frac{\partial f}{\partial x_1} + \delta\frac{\partial f}{\partial x_2} \tag{3}$$

and

$$\frac{\partial \Phi}{\partial X_1} = \alpha\frac{\partial \phi}{\partial x_1} + \gamma\frac{\partial \phi}{\partial x_2}, \qquad \frac{\partial \Phi}{\partial X_2} = \beta\frac{\partial \phi}{\partial x_1} + \delta\frac{\partial \phi}{\partial x_2}. \tag{4}$$

Hence multiplying the Jacobian determinant by D[1]

$$\begin{vmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \\ \frac{\partial \phi}{\partial x_1} & \frac{\partial \phi}{\partial x_2} \end{vmatrix} \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \begin{vmatrix} \alpha\frac{\partial f}{\partial x_1} + \gamma\frac{\partial f}{\partial x_2} & \beta\frac{\partial f}{\partial x_1} + \delta\frac{\partial f}{\partial x_2} \\ \alpha\frac{\partial \phi}{\partial x_1} + \gamma\frac{\partial \phi}{\partial x_2} & \beta\frac{\partial \phi}{\partial x_1} + \delta\frac{\partial \phi}{\partial x_2} \end{vmatrix} = \begin{vmatrix} \frac{\partial F}{\partial X_1} & \frac{\partial F}{\partial X_2} \\ \frac{\partial \Phi}{\partial X_1} & \frac{\partial \Phi}{\partial X_2} \end{vmatrix}. \text{ Q. E. D.}$$

[1] According to the rule for determinants of the same order: The element of the ith row and jth column of the product is formed by multiplying the elements of the ith row (or column) of either by the corresponding elements of the jth row (or column) of the other and adding the results. We have thus four options due to the interchangeability of rows and columns but whichever is chosen must of course be followed consistently throughout a particular multiplication.

Again *the Hessian of a binary form is a covariant.*

The direct proof of this theorem is similar to that employed in the case of the Jacobian. Let the original form f and its Hessian h be transformed by (1) into F and H. Then multiplying the Hessian of f by D (column by column and rearranging the differential operators in each element)

$$hD = \begin{vmatrix} \dfrac{\partial^2 f}{\partial x_1{}^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \dfrac{\partial^2 f}{\partial x_2{}^2} \end{vmatrix} \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} =$$

$$\begin{vmatrix} \dfrac{\partial}{\partial x_1}\left(\alpha \dfrac{\partial f}{\partial x_1} + \gamma \dfrac{\partial f}{\partial x_2}\right) & \dfrac{\partial}{\partial x_1}\left(\beta \dfrac{\partial f}{\partial x_1} + \delta \dfrac{\partial f}{\partial x_2}\right) \\ \dfrac{\partial}{\partial x_2}\left(\alpha \dfrac{\partial f}{\partial x_1} + \gamma \dfrac{\partial f}{\partial x_2}\right) & \dfrac{\partial}{\partial x_2}\left(\beta \dfrac{\partial f}{\partial x_1} + \delta \dfrac{\partial f}{\partial x_2}\right) \end{vmatrix} =$$

$$\begin{vmatrix} \dfrac{\partial}{\partial x_1} \dfrac{\partial F}{\partial X_1} & \dfrac{\partial}{\partial x_1} \dfrac{\partial F}{\partial X_2} \\ \dfrac{\partial}{\partial x_2} \dfrac{\partial F}{\partial X_1} & \dfrac{\partial}{\partial x_2} \dfrac{\partial F}{\partial X_2} \end{vmatrix} \text{ (from (2) and (3)).}$$

Now multiplying the last determinant (column by column) by D

$$hD^2 = \begin{vmatrix} \left(\alpha \dfrac{\partial}{\partial x_1} + \gamma \dfrac{\partial}{\partial x_2}\right) \dfrac{\partial F}{\partial X_1} & \left(\beta \dfrac{\partial}{\partial x_1} + \delta \dfrac{\partial}{\partial x_2}\right) \dfrac{\partial F}{\partial X_1} \\ \left(\alpha \dfrac{\partial}{\partial x_1} + \gamma \dfrac{\partial}{\partial x_2}\right) \dfrac{\partial F}{\partial X_2} & \left(\beta \dfrac{\partial}{\partial x_1} + \delta \dfrac{\partial}{\partial x_2}\right) \dfrac{\partial F}{\partial X_2} \end{vmatrix} =$$

$$\begin{vmatrix} \dfrac{\partial^2 F}{\partial X_1{}^2} & \dfrac{\partial^2 F}{\partial X_1 \partial X_2} \\ \dfrac{\partial^2 F}{\partial X_1 \partial X_2} & \dfrac{\partial^2 F}{\partial^2 X_2{}^2} \end{vmatrix} = H. \quad \text{Q. E. D.}$$

106. A method of deriving invariants.—Recurring to the results of §105 we saw that if f is transformed into F by

$$\begin{aligned} x_1 &= \alpha X_1 + \beta X_2 \\ x_2 &= \gamma X_1 + \delta X_2 \end{aligned} \tag{1}$$

then

$$\begin{aligned} \frac{\partial F}{\partial X_1} &= \alpha \frac{\partial f}{\partial x_1} + \gamma \frac{\partial f}{\partial x_2} \\ \frac{\partial F}{\partial X_2} &= \beta \frac{\partial f}{\partial x_1} + \delta \frac{\partial f}{\partial x_2}. \end{aligned} \tag{2}$$

Solving these equations we may write

$$\begin{aligned} D \frac{\partial f}{\partial x_2} &= \alpha \frac{\partial F}{\partial X_2} + \beta \left(-\frac{\partial F}{\partial X_1}\right) \\ D \left(-\frac{\partial f}{\partial x_1}\right) &= \gamma \frac{\partial F}{\partial X_2} + \delta \left(-\frac{\partial F}{\partial X_1}\right). \end{aligned} \tag{3}$$

Comparing equations (3) with (1) we observe that except for the factor D the operating symbols $\partial/\partial x_2, -\partial/\partial x_1$ go into $\partial/\partial X_2, -\partial/\partial X_1$ by precisely the same transformation that carries x_1, x_2 into X_1, X_2. We have in fact, f being of order n, not only

$$F(X_1, X_2) = f(x_1, x_2) \tag{4}$$

but

$$F\left(\frac{\partial}{\partial X_2}, -\frac{\partial}{\partial X_1}\right) = D^n f\left(\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1}\right). \tag{5}$$

If now ϕ is a covariant of order m so that

$$\phi(A_0, A_1 \ . \ . \ . \ X_1, X_2) = D^r \phi(a_0, a_1 \ . \ . \ . \ x_1, x_2) \tag{6}$$

then in virtue of (5) we obtain the equality of differential symbols

$$\phi\left(A_0, A_1 \ . \ . \ . \ \frac{\partial}{\partial X_2}, -\frac{\partial}{\partial X_1}\right) = D^{r+m} \phi\left(a_0, a_1 \ . \ . \ . \ \frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1}\right). \tag{7}$$

Then if we operate with (7) on (4), thus

$$\phi\left(A_0, A_1 \ . \ . \ . \ \frac{\partial}{\partial X_2}, -\frac{\partial}{\partial X_1}\right) F = D^{r+m} \phi\left(a_0, a_1 \ . \ . \ . \ \frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1}\right) f, \tag{8}$$

we conclude that the function on the right is a covariant since the two sides are identical except for a power of D. Or we may summarize:

If in any covariant ϕ *of* f *we replace* x_1, x_2 *by the differential symbols* $\partial/\partial x_2, -\partial/\partial x_1$ *and operate on* f *we obtain another covariant.*

But this is just the polar process (§98) and the theorem may be otherwise stated

If ϕ *is a covariant of* f, *the polar of* ϕ *with respect to* f *is also a covariant, unless indeed it vanish when* f *and* ϕ *are apolar.*

By exactly the same reasoning the foregoing result can be extended to any two binary forms thus

If ϕ^m *and* f^n *are binary forms of orders* m *and* n, $m \leqq n$, *the polar of* ϕ *with respect to* f *is a covariant of the forms.*

This is the theorem of Boole,[1] one of the earliest as it is one of the most important methods of generating covariants.

As a corollary, *the condition*[2] *that two forms of the same order be apolar, or that a form be self-apolar is an invariant.*

Again referring to equations (1) and (3) it appears that, apart from the constant D, the polar forms $\partial f/\partial x_2, -\partial f/\partial x_1$ are transformed in the same manner as the variables. It follows that if $\phi(x_1\ x_2)$ *is a covariant of* f *(including* f *itself) then the function of first polars* $\phi(\partial f/\partial x_2, -\partial f/\partial x_1)$ *is another covariant*, a theorem due to Sylvester.

This covariant which is of order $m(n-1)$ will always contain f as a factor.[3] The new covariant is thus of order $mn - m - n$ and when $\phi \equiv f$ of order $n^2 - 2n$.

107. Complete systems of concomitants.—The first problem that confronted the founders of the invariant theory was the discovery of particular functions and types

[1] A pioneer worker in the invariant theory, his first paper having appeared, *Cambridge Mathematical Journal*, Nov., 1841.

[2] Equations (1) and (2), §99.

[3] See for example Elliott, *Algebra of Quantics*, §50, Ex. 8.

of functions which possess the invariant property. We are now in a position to see how a system of invariant forms can be constructed. Beginning with a single binary form f, we saw that the discriminant is an invariant. So also is the condition for self-apolarity if the form be even. Likewise the Hessian H is a covariant and the Jacobian J of f and H is another. Next the polar of J with respect to f will be a covariant. Continuing, discriminants, self-apolarity conditions and Hessians of the various covariants will be new invariants and covariants. Still more invariant forms can be found by taking the Jacobian of any two covariants and the polar of any covariant with respect to any other. It must not be expected however that all the forms so derived will be distinct,—and some will vanish identically. On the other hand there might be forms which these processes fail to reveal.

The question is whether there is any limit to the number of distinct concomitants, *i. e.*, whether there is a *complete system* of such forms? While in a sense the number is infinite, it can be proved without difficulty[1] that the number of *independent* invariants and covariants (including f itself) cannot exceed n, the order of f,—is in fact precisely equal to n. And the number of independent absolute invariants alone is equal to $n - 3$, *i. e.*, the order of f diminished by the number of essential constants in the linear transformation. This system of n independent forms is called an *algebraically* complete system since all other forms can be expressed algebraically in terms of them.

But the classical theory is concerned with rational, integral invariants and covariants only and the corresponding question is whether there is a fundamental system, finite in number, of independent concomitants in terms of which all others are rationally and integrally expressible. And it

[1] Elliott, §42.

is such a system which is traditionally referred to as a complete system. The existence of a finite complete system in this sense was first established by Gordan with the aid of earlier theorems of Clebsch and is known as the *Clebsch-Gordan theorem.* The proof is too long and difficult to give here but may be found in Clebsch,[1] Grace and Young,[2] or Dickson.[3] Gordan also proved that any finite system of binary forms have a finite complete system of invariants and covariants, a theorem which was extended by Hilbert to algebraic forms in any number of variables.

While Hilbert's proof establishes only the existence of a complete system Gordan's furnishes at the same time a means of constructing it. But without resorting to Gordan's method we shall be able to obtain complete systems for several of the forms of lower order and these will be exhibited in later sections. Before proceeding to that topic however we shall in the immediate sequel make some important applications of the invariant theory to the binary cubic and quartic.

In the meantime the student is cautioned against the supposition that no interest attaches to invariants and covariants which do not belong to the fundamental system. For many projective properties give rise to invariants not included among the forms of the complete system although expressible in terms of them. Again when we are concerned with the interpretation of invariants as in geometry the fact that an invariant is expressible in terms of others frequently gives no clue whatever to its meaning.

EXERCISES

1. The Hessian of a binary form is a simultaneous covariant of the first polars $\partial f/\partial x_1$, $\partial f/\partial x_2$.

[1] *Theorie der binären algebraischen Formen*, sechster Abschnitt.
[2] *Algebra of Invariants*, Chap. 6.
[3] *Algebraic Invariants*, §§46–51.

2. The Jacobian $J(f, H)$ of a form and its Hessian is a covariant of f of degree 3 and order $3n - 6$.

3. Hence find the $C_{3,3}$ of the cubic $ax_1^3 + 3bx_1^2x_2 + 3cx_1x_2^2 + dx_2^3$.

4. Find the covariant of Ex. 3 by the theorem of Sylvester (§106).

5. Calculate the Hessian of the quartic $ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + y^4$. Then show that the polar of the quartic with respect to its Hessian is a numerical multiple of the catalecticant (§100). Therefore the catalecticant is an invariant.

6. Every form of even order has an invariant of the second degree,—the condition for self-apolarity. Recall this invariant for the quadratic, quartic and sextic.

7. The condition that two quadratics be apolar (harmonic) is a simultaneous invariant of the forms. Write the invariant.

108. Canonical forms.—In the binary domain just as in the ternary and quaternary[1] a general form can always be simplified by a suitable choice of the reference scheme. Thus any quadratic $q \equiv ax_1^2 + bx_1x_2 + cx_2^2$ (whose discriminant does not vanish) can be reduced to the form x_1x_2. In fact the reduction can be accomplished in a single infinity of ways. For it is only necessary to select as reference points the two points q while the choice of the unit point is unrestricted.

Otherwise, the equation of transformation of coördinates $\dfrac{x_1'}{x_2'} = \dfrac{ax_1 + bx_2}{cx_1 + dx_2}$ (§§47, 48) contains three parameters two only of which are specified if we merely ask that q take the form x_1x_2.

Again the general cubic

$$f \equiv ax_1^3 + bx_1^2x_2 + cx_1x_2^2 + dx_2^3$$

can be reduced to the canonical[2] form $X_1^3 + X_2^3$. For since the coördinates of three points may be assigned at random under the transformation above, the coördinates

[1] Familiar examples are the standard equations of the conics and quadrics.

[2] See definition §82.

of the points f may be taken as $X_1 : X_2 = -1 : -\omega : -\omega^2$ which effects the reduction.

This is not however a convenient method of actually performing the reduction since it requires first the solution of the cubic. On the contrary we shall show how the reduction can be effected in a way which leads to a

Solution of the cubic equation. If the coördinate system be changed so that the canonizant quadratic assumes the form X_1X_2 the cubic becomes $AX_1^3 + DX_2^3$ (Ex. 9, §100). This only fixes the reference points 0, ∞. Then writing $X_1 = X_1'/\sqrt[3]{A}$, $X_2 = X_2'/\sqrt[3]{D}$, which fixes the unit point, the reduction is completed.

It thus appears that the variables in the transformed (canonical) equation of the cubic are, except for a constant multiple, simply the linear factors of the canonizant.[1] And since the transformed cubic is readily solved we can by reversing the transformation obtain the roots of the original. Or we may proceed as in the following

Example. To solve the cubic

$$f \equiv 11x^3 - 3x^2 + 15x + 2 = 0$$

first calculate the canonizant which is

$$H = 2x^2 + x - 1 = (2x - 1)(x + 1).$$

Hence the new variables are

$$X_1 = 2x - 1, X_2 = x + 1$$

and we may write

$$f \equiv A(2x - 1)^3 + D(x + 1)^3.$$

Equating coefficients of like terms in this identity

$$8A + D = 11, \qquad -A + D = 2,$$

[1] Hence the origin of the name *canonizant*.

whence $A = 1$, $D = 3$. The roots of the cubic are given therefore by

$$2x - 1 + (x + 1)\sqrt[3]{3} = 0,$$

where all three values of the radical are to be used.

109. Remarks on canonical forms.—In considering whether a binary quantic can be made to assume a particular form under a linear, *i. e.*, a projective transformation, the first question is does the proposed form contain a sufficient number of constants? For if two forms are to be projectively equivalent they must contain explicitly or implicitly the same number of essential constants. Thus the general quadratic contains three homogeneous constants but the form X_1X_2 contains implicitly four since each factor represents a linear function like $\alpha x_1 + \beta x_2$. And the reduction to this form is possible in an infinity of ways only because the new form contains one constant more than the old. Again the general cubic and the canonical form selected contain each four homogeneous constants.

To put it differently, the general collineation contains three essential constants and these can be employed to specify a maximum of three of the essential constants in the original form. In other words the number of essential constants that explicitly remain must be at least equal to the number of absolute invariants of the original form.[1] It does not follow that we can assign arbitrary values to *any* three constants in the original form since in so doing we might impose a projective condition on the form. Thus in the case of the cubic in the preceding section we were able to choose the coefficients so that $a{:}b{:}c{:}d = 1{:}0{:}0{:}1$. An alternative choice would be $a{:}b{:}c{:}d = 0{:}1{:}1{:}0$ for that is merely choosing two roots of the cubic for reference points and the third to be -1. We cannot however transform

[1] These are the reasons in fact which fix the number of absolute invariants in the binary domain at $n - 3$.

the general cubic so that $c = d = 0$, for that would mean that the transformed cubic had equal roots,—a projective property not belonging to the original and projective properties must not be molested.

We see therefore that it is necessary but not sufficient that the new form contain the proper number of constants. We must make sure that the projective properties of the two forms are identical and this is not a matter always easy to determine in advance. Generally speaking it is possible to divide out one coefficient, choose two of the coefficients (except the first two or the last two) to be zero and a fourth to be 1. But each case should be examined for itself.

The importance of canonical forms in the invariant theory depends on the fact that any invariant relation which holds for the canonical form holds equally for the general form. Hence to discover or verify these relations it is sufficient to work with the simplest form available, an obvious saving of labor. Moreover the geometric interpretation of concomitants is greatly facilitated, sometimes rendered obvious, by the use of canonical forms.

110. Canonical forms of the quartic.—Let the quartic $f \equiv ax_1^4 + 4bx_1^3x_2 + 6cx_1^2x_2^2 + 4dx_1x_2^3 + ex_2^4$ be changed by a linear transformation into $F \equiv AX_1^4 + 4BX_1^3X_2 + 6CX_1^2X_2^2 + 4DX_1X_2^3 + EX_2^4$. We have seen that the quartic has two invariants,—the apolarity condition and the catalecticant which may be designated by I_2 and I_3 and whose values are $I_2 \equiv ae - 4bd + 3c^2$ and $I_3 \equiv ace + 2bdc - c^3 - b^2e - ad^2$.

The catalectic quartic. We first show that *if f have an apolar quadratic, i. e., if the catalecticant vanish, it can be reduced to the canonical form $X_1^4 + X_2^4$.* For if a quadratic be apolar to a quartic the relation will persist after a linear transformation.[1] Hence if we apply to the quartic a trans-

[1] This is another proof that the catalecticant is an invariant.

formation that reduces the apolar quadratic to X_1X_2, the quartic will assume the form $AX_1^4 + EX_2^4$ since if X_1X_2 is apolar to F, $B = C = D = 0$. Then writing $X_1 = X_1'/\sqrt[4]{A}$, $X_2 = X_2'/\sqrt[4]{E}$ the problem is solved.

The canonical form can be resolved into the apolar quadratic factors $X_1^2 + iX_2^2$ and $X_1^2 - iX_2^2$. We have thus a geometrical interpretation of the catalecticant:

The necessary and sufficient condition that a quartic represent harmonic pairs of points is the vanishing of the catalecticant.

The general quartic. It follows from the foregoing or it may be inferred directly by a counting of constants that the general quartic cannot be reduced to the sum of two fourth powers by a linear transformation. On the other hand it can be reduced to the sum of three fourth powers in a single infinity of ways for $X^4 + Y^4 + Z^4$ contains implicitly six homogeneous constants or one more than necessary.

A natural canonical form is $X_1^4 + 6\lambda X_1^2X_2^2 + X_2^4$ which contains the requisite number of constants,—two each for the linear functions X_1 and X_2 and one explicit constant λ. We shall now show how the reduction to this form can be effected, proving *ipso facto* that the reduction is possible.

A preliminary problem to be solved is: *To find a quadratic whose polar with respect to the quartic f is to a constant factor the quadratic itself.*

If $q \equiv \alpha x_1^2 + \beta x_1x_2 + \gamma x_2^2$ be such a quadratic we ask that

$$\left(\gamma \frac{\partial^2}{\partial x_1^2} - \beta \frac{\partial^2}{\partial x_1 \partial x_2} + \alpha \frac{\partial^2}{\partial x_2^2}\right) f \equiv 12\, kq \tag{1}$$

or

$$\gamma(ax_1^2 + 2bx_1x_2 + cx_2^2) - \beta(bx_1^2 + 2cx_1x_2 + dx_2^2) + \alpha(cx_1^2 + 2dx_1x_2 + ex_2^2) \equiv k(\alpha x_1^2 + \beta x_1x_2 + \gamma x_2^2).$$

Equating coefficients we obtain the system of equations

$$\begin{aligned} a\gamma - b\beta + c\alpha &= k\alpha \\ b\gamma - c\beta + d\alpha &= \tfrac{1}{2}k\beta \\ c\gamma - d\beta + e\alpha &= k\gamma \end{aligned} \qquad (2)$$

whose consistency requires

$$\begin{vmatrix} a & b & c - k \\ b & c + \frac{1}{2}k & d \\ c - k & d & e \end{vmatrix} = 0, \qquad (3)$$

a cubic in k. Hence there are three quadratics of the kind sought,—one corresponding to each value of k in equation (3).

Now replacing x_1, x_2 by the linear factors of q the quartic is brought to the canonical form. If $q \equiv (\alpha_1 x_1 + \alpha_2 x_2)(\beta_1 x_1 + \beta_2 x_2) \equiv X_1 X_2$ then we are to establish the identity

$$f \equiv a x_1^4 + \ . \ . \ . \ \equiv A(\alpha_1 x_1 + \alpha_2 x_2)^4 + 6k'q^2 + E(\beta_1 x_1 + \beta_2 x_2)^4. \qquad (4)$$

Operating on both sides of (4) with q we find

$$\left(\gamma \frac{\partial^2}{\partial x_1^2} - \beta \frac{\partial^2}{\partial x_1 \partial x_2} + \alpha \frac{\partial^2}{\partial x_2^2}\right) f = 24(4\alpha\gamma - \beta^2)k'q \text{ or } 12kq \quad (5)$$

where

$$k = 2(4\alpha\gamma - \beta^2)k'.$$

The calculation on the right is facilitated by observing that the operating symbol can be written in the alternative forms

$$\left(\gamma \frac{\partial^2}{\partial x_1^2} - \beta \frac{\partial^2}{\partial x_1 \partial x_2} + \alpha \frac{\partial^2}{\partial x_2^2}\right) \equiv \left(\alpha_1 \frac{\partial}{\partial x_2} - \alpha_2 \frac{\partial}{\partial x_1}\right)\left(\beta_1 \frac{\partial}{\partial x_2} - \beta_2 \frac{\partial}{\partial x_1}\right)$$

and that (§100, Ex. 21)

$$\left(\alpha_1 \frac{\partial}{\partial x_2} - \alpha_2 \frac{\partial}{\partial x_1}\right)(\alpha_1 x_1 + \alpha_2 x_2)^4 \equiv 0,$$

and

$$\left(\beta_1 \frac{\partial}{\partial x_2} - \beta_2 \frac{\partial}{\partial x_1}\right)(\beta_1 x_1 + \beta_2 x_2)^4 \equiv 0,$$

so that we get nothing by operating with q on the first and third terms while q on $q^2 = 2q \cdot 2\Delta$.

Thus the identity (5) will hold provided we choose k to satisfy (3), since (5) is the same as (1) and leads to the same cubic in k. In other words f has been reduced to the form $AX_1^4 + 6k'X_1^2X_2^2 + EX_2^4$ when A and E can be removed as before. Summarizing

1°. *There are three pairs of points, represented by the quadratics q, which are their own polars with respect to a set of four points f.*

2°. *A general quartic f can be reduced to the canonical form in three ways,* viz., *by taking for new variables the linear factors of the three quadratics q.*

To expand equation (3) which is called the *resolvent cubic* of the quartic, note that if $k = 0$ the determinant reduces to I_3, *i. e.*, the term free of k is the invariant I_3. Changing signs and clearing of fractions the equation is found to be

$$k^3 - I_2k - 2I_3 = 0. \tag{6}$$

111. Solution of the quartic.—Obviously the reduction of a quartic to the canonical form leads to its solution for the reduced equation has the form of a quadratic. The method is perhaps sufficiently indicated by the preceding paragraph but an example will make it plain. Let us solve the equation

$$f \equiv 2x^4 + 16x^3 + 12x^2 + 28x + 23 = 0.$$

Calculating the invariants, $I_2 = -54$, $I_3 = -270$ whence the resolvent cubic is

$$k^3 + 54k + 540 = 0$$

one root of which is -6. Substituting this value for k in

the first two equations (2), §110, we have to determine α, β, γ

$$8\alpha - 4\beta + 2\gamma = 0$$
$$7\alpha + \beta + 4\gamma = 0.$$

Hence

$$\alpha : \beta : \gamma = -18 : -18 : 36$$

i. e.,

$$q = x^2 + x - 2 = (x + 2)(x - 1)$$

and the new variables are

$$X_1 = x + 2, \qquad X_2 = x - 1.$$

It follows that the quartic can be written in the identical forms

$$f \equiv A(x + 2)^4 + B(x^2 + x - 2)^2 + C(x - 1)^4.$$

Equating coefficients

$$A + B + C = 2$$
$$16A + 4B + C = 23$$
$$8A + 2B - 4C = 16$$

whence $A = 1$, $B = 2$, $C = -1$ and the canonical form is

$$(x + 2)^4 + 2(x + 2)^2(x - 1)^2 - (x - 1)^4.$$

Solving this as a quadratic the four roots of the quartic are given by

$$x + 2 = \pm\sqrt{-1 \pm \sqrt{2}}\,(x - 1).$$

EXERCISES

1. Show that a quadratic can be reduced to the canonical form $x^2 + y^2$. How many ways can this be done?

2. Prove that a quadratic with a double root can be reduced to the form x_1^2 or x_2^2; that a cubic with a double root has the canonical form $x_1^2x_2$ or $x_1x_2^2$.

3. Show that two quadratics can be reduced to the canonical forms $x^2 + y^2$, $x^2 + ky^2$. (Take as reference points the double points of the involution determined by the quadratics.)

4. Solve the cubic equations

$$(a)\ 9x^3 + 21x^2 + 27x + 7 = 0,$$
$$(b)\ 16x^3 + 12x^2 + 30x + 7 = 0,$$
$$(c)\ 3x^3 - 48x^2 + 6x - 94 = 0.$$

5. A second canonical form of the general cubic is $x_1^2x_2 - x_1x_2^2$. What is the canonizant of this cubic? Outline a method for reducing a cubic to this form.

6. The polar of each Hessian point with respect to a set of three points is the other Hessian point repeated. (Use the canonical form.)

7. A quartic with a double point can be reduced to the canonical form $x_1^2(x_1^2 + x_2^2)$; one with two double points to $x_1^2x_2^2$.

8. If a binary quartic have a double point the Hessian has the same double point. (Prove for the canonical form of Ex. 7.)

9. If a quartic f have two double points the Hessian has the same pair of double points and $f \equiv kH$ where k is a constant.

10. If a quartic have a triple point, the Hessian has the same point for a four-fold point. (Prove for $x_1^4 + 4x_1^3x_2$.)

11. Show that a general quartic can be reduced to the form $(x_1^3 + x_2^3)(x_1 + kx_2)$.

12. Solve the quartic $2x^4 - 8x^3 + 24x^2 + 8x + 1 = 0$.

13. Reduce to the canonical form $x^4 + 8x^3 - 12x^2 + 104x - 20 = 0$.

14. Solve the quartic $3x^4 - 4x^3 + 24x^2 - 16x + 48 = 0$.

15. The Hessian of a catalectic quartic has two square factors.

Elementary Complete Systems

112. We shall devote the remainder of this chapter to an enumeration and geometrical interpretation of complete systems[1] of invariants of forms up to the fourth order, most of these having already been encountered in some connection. The form itself, being an absolute covariant, is always included in the complete system. For proofs that the systems are complete, the student is referred to the treatises (§94, footnote).

Linear forms. One linear form possesses no interest, the complete system consisting of the form itself, while the only covariants of any kind are powers of the form.

[1] See §107.

Two linear forms $a_1x_1 + b_1x_2$ and $a_2x_1 + b_2x_2$ have in addition to the forms themselves an invariant, the resultant $a_1b_2 - a_2b_1$, whose vanishing signifies that the two forms represent the same point. And the complete system for any number of linear forms consists of the base forms and the resultants of each pair.

113. Concomitants of a system of quadratics.—*A single quadratic* $f_1 \equiv a_1x_1^2 + 2b_1x_1x_2 + c_1x_2^2$ has one invariant, the discriminant, which it is customary to write $a_1c_1 - b_1^2$ and whose vanishing is the necessary and sufficient condition that the quadratic be a square or represent two coincident points. The discriminant and the quadratic itself constitute the complete system of f_1.

Two quadratics f_1 and $f_2 \equiv a_2x_1^2 + 2b_2x_1x_2 + c_2x_2^2$ possess a simultaneous invariant $D_{12} \equiv a_1c_2 - 2b_1b_2 + a_2c_1$ whose vanishing expresses the condition that f_1 and f_2 represent apolar or harmonic pairs of points. When $f_1 \equiv f_2$ this invariant reduces to twice the discriminant, thus $D_{11} \equiv 2(a_1c_1 - b_1^2)$.

There is also a simultaneous covariant, the Jacobian of the two $J(f_1, f_2)$, which represents a pair of points harmonic with f_1, f_2 That is J gives the double points of the involution determined by the quadratics and it may be written (§86, Equation (5))

$$J_{12} \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ x_2^2 & -x_2x_1 & x_1^2 \end{vmatrix}$$
$$= |a\ b|x_1^2 + |a\ c|x_1x_2 + |b\ c|x_2^2.$$

This completes the fundamental system of two quadratics which thus consists of the six forms f_1, f_2, the two invariants[1] D_{11}, D_{22}, the apolarity invariant D_{12} and the covariant J.

[1] In the complete systems of two or more quadratics we shall, because of the convenience of notation, use D_{ii}, *i. e.*, *twice* the discriminants instead of the discriminants of the several forms.

The resultant R_{12} of f_1 and f_2 is a second joint invariant but it is not included in the complete system since it can be expressed in terms of the other forms. It will be instructive as an exemplification of the great Clebsch-Gordan principle to see how this can be done. Recalling (§102, Ex. 4) that the resultant of two quadratics is the discriminant of their Jacobian we have

$$R_{12} \equiv 4(a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1) - (a_1c_2 - a_2c_1)^2$$

which involves the coefficients of each form to the second degree. Now the only invariants of degree two in the coefficients of f_1 and f_2 which can be built out of the forms of the complete system are $D_{11}D_{22}$ and D_{12}^2. Hence R_{12} which is expressible rationally and integrally in terms of these two invariants must be a linear combination of them, thus

$$R_{12} \equiv \lambda D_{11}D_{22} + \mu D_{12}^2.$$

Equating coefficients we find $\lambda = 1$, $\mu = -1$, *i. e.*,

$$R_{12} \equiv 4(a_1c_1 - b_1^2)(a_2c_2 - b_2^2) - (a_1c_2 - 2b_1b_2 + a_2c_1)^2 = D_{11}D_{22} - D_{12}^2.$$

The identity just written is still more easily found by using the canonical forms of the quadratics (§111, Ex. 3). For if $f_1 = x^2 + y^2$ and $f_2 = ax^2 + cy^2$, the invariants are $D_{11} = 2$, $D_{22} = 2ac$, $D_{12} = a + c$, $R_{12} = -(a - c)^2$ and the relation is obvious.

It is clear from the second form of R_{12} that the resultant is also the discriminant of the quadratic $Q \equiv D_{11}x_1^2 + 2D_{12}x_1x_2 + D_{22}x_2^2$. *The vanishing of the resultant may therefore be interpreted variously as the condition that* (a) f_1 *and* f_2 *have a point in common,* (b) J *represent coincident points, i. e., the involution* $f_1 + kf_2 = 0$ *be parabolic (degenerate), or* (c) Q *represent coincident points.*

The complete system for a set of three quadratics f_1, f_2, and $f_3 \equiv a_3x_1^2 + 2b_3x_1x_2 + c_3x_2^2$ includes the fundamental

forms of the quadratics taken in pairs, consisting of the three base forms, their three double discriminants, three apolarity invariants and three Jacobians. There is however a new invariant

$$I_{123} \equiv |a\ b\ c| \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

whose vanishing expresses the condition that the three pairs of points f belong to an involution (§87).

Three quadratics find their best geometric interpretation in the theory of the conic as a rational curve (post, Chap. X).

Four quadratics present nothing new since any quadratic whatever can be written as a linear function of f_1, f_2, f_3, (§99). The complete system is made up of the complete systems of the base forms considered three at a time.

114. The complete system of the cubic consists of four forms,—all familiar. They are the cubic itself, a $C_{1,3}$, the Hessian H, a $C_{2,2}$, the Jacobian $J(f, H)$ of the two, a $C_{3,3}$ called the cubic covariant, and the discriminant Δ, a $C_{4,0}$. The discriminant is best found as the discriminant, with sign changed, of the Hessian with which it is identical (§102, Ex. 3). The fundamental forms are

$$f\colon ax_1^3 + 3bx_1^2x_2 + 3cx_1x_2^2 + dx_2^3$$
$$H\colon (ac - b^2)x_1^2 + (ad - bc)x_1x_2 + (bd - c^2)x_2^2$$
$$\begin{aligned} J\colon &(a^2d - 3abc + 2b^3)x_1^3 + 3(abd + b^2c - 2ac^2)x_1^2x_2 \\ &+ 3(2b^2d - acd - bc^2)x_1x_2^2 + (3bcd - ad^2 - 2c^3)x_2^3 \end{aligned}$$
$$\begin{aligned} \Delta\colon &(ad - bc)^2 - 4(ac - b^2)(bd - c^2) = \\ &a^2d^2 + 4ac^3 + 4b^3d - 3b^2c^2 - 6abcd. \end{aligned}$$

Since a cubic cannot have more than three independent invariants (§107) there must be an identical relation connecting the fundamental forms,—a relation easily found with the help of the canonical form. For if $f = ax^3 + dy^3$,

we have $\Delta = a^2d^2$, $H = adxy$, $J = ad(ax^3 - dy^3)$, whence immediately

$$\Delta f^2 - J^2 \equiv 4H^3. \qquad (1)$$

A relation like (1) among the fundamental forms is called a *syzygy*. The existence of syzygies is not surprising in view of the requirement for rational and integral independence of the forms of the complete system. In fact syzygies occur in the complete systems of all forms except that of the quadratic.

N.B.—When investigating syzygies or other invariant relations by means of canonical forms it is well to give each term a literal coefficient. For in any invariant identity each term must be of the same degree in the coefficients (as well as the variables). And it is easier to make sure that this is the case when the coefficients are letters. Suppose, *e. g.*, that in seeking the syzygy just established we had taken $f = x^3 + y^3$, then $\Delta = 1$, $H = xy$, $J = x^3 - y^3$. While for these forms $f^2 - J^2 \equiv 4H^3$, this is not a valid invariant identity since J^2 and H^3 are of degree 6 and order 6 but f^2 is a $C_{2,6}$. It is therefore necessary to throw in the proper (first) power of Δ a $C_{4,0}$ to bring f^2 to the right degree.

Caution must also be observed in regard to numerical coefficients and algebraic signs. In making up the complete system the fundamental forms are chosen with a definite convention as to sign and numerical coefficient and these conventions must not be violated. Thus some writers take for the discriminant of the quadratic (§113) D_{11}, for the discriminant of the cubic -2Δ, etc.

115. A geometrical interpretation of the system of the cubic appears at once from the canonical form. If $f \equiv x^3 - 1$, $J \equiv x^3 + 1$ and $H \equiv x$, hence $f = 0$ represents the points $1, \omega, \omega^2$, $J = 0$ the points $-1, -\omega, -\omega^2$ and $H = 0$

the points $0, \infty$. Now the polar (harmonic conjugate) of each point of f with respect to the other two is a point of J, *e. g.*, the polar of 1 with respect to ω, ω^2 is -1. We obtain thus the three pairs $\pm 1, \pm\omega, \pm\omega^2$ each of which is harmonic with the pair $0, \infty$. Or if the points f be denoted by $\alpha_1, \alpha_2, \alpha_3$, and the points J by $\beta_1, \beta_2, \beta_3$ we may say

The polar of α_i as to α_j, α_k, $i \neq j \neq k$, is a point β_i which forms with α_i a conjugate pair in the quadratic involution whose double points are the points H.

In fact the relation between the points f and J is reciprocal and the theorem can be stated with α_i and β_i interchanged.

The syzygetic pencil of binary cubics.

Again consider the cubic involution determined by a cubic f and its cubicovariant J

$$\lambda f + \mu J = 0, \tag{1}$$

which we shall call a *syzygetic pencil*, and which represents geometrically a single infinity of triads of points. The properties of the involution can be studied most conveniently by taking (1) in the canonical form. If now f be reduced to the form $f' \equiv X^3 + Y^3$, the cubicovariant becomes $J' = X^3 - Y^3$, hence the involution can be written

$$\lambda f' + \mu J' \equiv \lambda(X^3 + Y^3) + \mu(X^3 - Y^3) \equiv \lambda' X^3 + \mu' Y^3. \tag{2}$$

In this form X and Y are simply the factors of the Hessian of f'. Therefore, H being a covariant,

The syzygetic pencil (1) *can be written in the canonical form $\lambda' X^3 + \mu' Y^3$ if we take for new variables the linear factors of the Hessian of f.*

The double points of the involution (2) are given by the Jacobian of f' and J' (§102, 3°) which is found to be $-2X^2Y^2 = -2H'^2$. Hence instead of four sets with double

points as in the general case we have but two (repeated) sets. To find the sets which have double points we need only write the discriminant of (2) which is $\lambda'^2\mu'^2$. If this vanish we obtain the sets X^3 and Y^3 which are thus cubes of the Hessian factors. Therefore if two points of a set coincide all three coincide forming a triple point. In view of the invariant nature of the results we may say

Among the cubics of the syzygetic pencil (1) *there are two and only two with repeated factors and these are perfect cubes of the factors of the Hessian.* Or in other words, *the involution has two and only two multiple points, namely the triple points given by* $H = 0$.

Recalling the syzygy $\Delta f^2 - J^2 \equiv 4H^3$, it is clear that since the right side is a perfect third power the left side and its factors as well must be also. Thus the two cubes in the pencil are precisely $J + \sqrt{\Delta}f$ and $J - \sqrt{\Delta}f$.

It is interesting to observe that if u is any cubic of the syzygetic pencil, then u' its cubicovariant also belongs to the pencil. For (equation (2)) if $u = \lambda' X^3 + \mu' Y^3$, $u' = \lambda'\mu'(\lambda' X^3 - \mu' Y^3)$. Moreover the cubicovariant of u' is to a factor the original cubic u. Thus the relation between u and u' is mutual or involutory. In other words

The sets of the cubic involution separate into conjugate pairs u and u' of a quadratic involution.[1]

In the quadratic involution there are two sets which are self-conjugate. These are found by asking that the coefficients of u and u' be proportional, *i. e.*, $\lambda'/\lambda'^2\mu' = -\mu'/\lambda'\mu'^2$ or $2\lambda'^2\mu'^2 = 0$. Hence *the self-conjugate* (*double*) *sets are the Hessian points taken three times.*

We are thinking here of the cubic geometrically as

[1] The point is the element in the cubic involution which consists of ∞^1 triads of points such that one point of a triad determines the other two points of the same triad. In the quadratic involution on the other hand the elements are triads and the involution is made up of the single infinity of pairs of triads u and u' which uniquely determine each other.

representing three points of a range or three lines of a pencil. But the theorems considered abstractly apply to the whole binary domain and will enable us therefore to supply other interpretations when the cubic represents points on a conic or other rational curve.[1]

EXERCISES

Exercises 1–17 refer to one or more quadratic forms where the notation is the same as in §113.

1. Show that the discriminant of $f_1 + \lambda f_2$ considered as a single quadratic is a quadratic in λ every coefficient of which is an invariant. The discriminant of the quadratic in λ is R_{12}.

2. The pencil $f_1 + \lambda f_2 = 0$ can be written in the form $X_1^2 + kX_2^2 = 0$ if the new variables are taken as factors of the quadratic apolar to f_1 and f_2. (*Cf.* Ex. 6, §89.)

3. Show that the condition that f_1, f_2 and J_{12} belong to an involution is the vanishing of the eliminant of f_1 and f_2. (Identify R_{12} with the invariant I_{123} of the system f_1, f_2, J_{12}.)

4. By considering the degree in the coefficients of the two forms and the order in the variables, find an expression for the square of the Jacobian of two quadratics in terms of the complete system. (The canonical forms $x^2 + y^2$ and $ax^2 + by^2$ may be used.) *Ans.* $-2J_{12}^2 \equiv D_{22}f_1^2 - 2D_{12}f_1f_2 + D_{11}f_2^2$.

5. Obtain the relation of Ex. 4 by multiplying the determinants: (See footnote §105.)

$$2J_{12}^2 \equiv \begin{vmatrix} x_2^2 & -x_1x_2 & x_1^2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \quad \begin{vmatrix} x_1^2 & 2x_1x_2 & x_2^2 \\ c_1 & -2b_1 & a_1 \\ c_2 & -2b_2 & a_2 \end{vmatrix}.$$

6. Establish the identity $f_1J_{23} + f_2J_{31} + f_3J_{12} \equiv 0$.

7. If three quadratics are harmonic in pairs they can be written in the canonical form xy, $x^2 + y^2$, $x^2 - y^2$.

8. If three quadratics f_1, f_2, f_3 are mutually apolar there is an identical relation connecting their squares $k_1f_1^2 + k_2f_2^2 + k_3f_3^2 \equiv 0$, where the k's are constants. (Prove for the canonical form of Ex. 7 and linearly transform the system. Or since each quadratic gives the double points of the involution determined by the other two, we must have $f_i \equiv \lambda_iJ_{jk}$. Then substitute in 6.)

[1] See §§133, 169.

9. Prove the identity for four quadratics

$$2J_{12}J_{34} \equiv \begin{vmatrix} x_2^2 & -x_1x_2 & x_1^2 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \begin{vmatrix} x_1^2 & 2x_1x_2 & x_2^2 \\ c_3 & -2b_3 & a_3 \\ c_4 & -2b_4 & a_4 \end{vmatrix} \equiv \begin{vmatrix} 0 & f_1 & f_2 \\ f_3 & D_{13} & D_{23} \\ f_4 & D_{14} & D_{24} \end{vmatrix}.$$

10. Prove the identity

$$0 \equiv \begin{vmatrix} a_1 & b_1 & c_1 & f_1 \\ a_2 & b_2 & c_2 & f_2 \\ a_3 & b_3 & c_3 & f_3 \\ a_4 & b_4 & c_4 & f_4 \end{vmatrix} \equiv f_1I_{234} - f_2I_{341} + f_3I_{412} - f_4I_{123}.$$

11. By multiplying determinants establish the identity connecting three quadratics and their invariants:

$$0 \equiv \begin{vmatrix} a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \\ a_3 & b_3 & c_3 & 0 \\ x_2^2 & -x_1x_2 & x_1^2 & 0 \end{vmatrix} \begin{vmatrix} c_1 & -2b_1 & a_1 & 0 \\ c_2 & -2b_2 & a_2 & 0 \\ c_3 & -2b_3 & a_3 & 0 \\ x_1^2 & 2x_1x_2 & x_2^2 & 0 \end{vmatrix} \equiv \begin{vmatrix} D_{11} & D_{12} & D_{13} & f_1 \\ D_{21} & D_{22} & D_{23} & f_2 \\ D_{31} & D_{32} & D_{33} & f_3 \\ f_1 & f_2 & f_3 & 0 \end{vmatrix}.$$

12. From Ex. 11 derive the invariant identity connecting the squares of three mutually apolar quadratics.

13. Verify the identity for six quadratics:

$$2I_{123}I_{456} \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} c_4 & -2b_4 & a_4 \\ c_5 & -2b_5 & a_5 \\ c_6 & -2b_6 & a_6 \end{vmatrix} \equiv \begin{vmatrix} D_{14} & D_{15} & D_{16} \\ D_{24} & D_{25} & D_{26} \\ D_{34} & D_{35} & D_{36} \end{vmatrix}.$$

14. Hence and by §52

$$2\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \equiv 2I_{123}^2 \equiv \begin{vmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{vmatrix}$$

where A_i are the cofactors of a_i in I_{123}.

15. From their determinant expressions prove

$$2J_{jk}I_{ijk} \equiv \begin{vmatrix} f_i & f_j & f_k \\ D_{ji} & D_{jj} & D_{jk} \\ D_{ki} & D_{kj} & D_{kk} \end{vmatrix}.$$

16. Verify the following invariant relations which express concomitants of a system of quadratics in terms of the fundamental forms, where $D_{12,3}$ is the apolarity invariant of J_{12} and f_3, $I_{1,2,34}$ is the involution invariant for f_1, f_2, J_{34}, etc.:

(*a*) $D_{12,3} \equiv I_{123}$

(*b*) $J_{12,3} \equiv f_2D_{13} - f_1D_{23}$

(*c*) $J_{12,34} \equiv f_2I_{134} - f_1I_{234}$. (Replace f_3 in (*b*) by J_{34} and apply (*a*), thus $J_{12,34} \equiv f_2D_{1,34} - f_1D_{2,34} \equiv f_2I_{134} - f_1I_{234}$.)

By interchanging 1 with 3 and 2 with 4 in (*c*), derive 10.

(*d*) $2I_{12,3,4} \equiv D_{13}D_{24} - D_{14}D_{23}$.

(*e*) $D_{12,34} \equiv I_{12,3,4}$.

(*f*) $2I_{12,34,5} \equiv D_{25}I_{134} - D_{15}I_{235}$. Replace f_3 by J_{34} and f_4 by f_5 in (*d*) and apply (*a*).

(*g*) $2I_{12,34,56} \equiv I_{256}I_{134} - I_{156}I_{234}$. Replace f_5 by J_{56} in (*f*) and apply (*a*).

17. The double ratio ρ of the two pairs of points f_1, f_2 is given by $\rho^2 - 2\rho \dfrac{D_{12}{}^2 + D_{11}D_{22}}{D_{12}{}^2 - D_{11}D_{22}} + 1 = 0.$

The remaining exercises refer to the cubic, the notation corresponding to that of §114.

18. If a cubic f have a double root, H has the same double root and J has the root for a triple root. Therefore the double root is given by $J/H = 0$ and the simple root by $f/H = 0$.

19. Hence solve the cubic with a double root:

$$18x^3 - 15\sqrt{2}\,x^2 - 8x + 8\sqrt{2} = 0.$$

20. If the cubic have a triple root, $H \equiv 0$, *i. e.*, $ac - b^2 = 0$, $ad - bc = 0$, $bd - c^2 = 0$, or $a/b = b/c = c/d$, and conversely.

21. By consideration of degree and order show that (*a*) the discriminant of J is a numerical multiple of Δ^3; (*b*) Hessian of J is a constant times ΔH. Verify these results by the canonical form.

22. Show that the cubicovariant may be obtained by operating on the discriminant as follows

$$\left(x_2{}^3 \frac{\partial}{\partial a} - x_2{}^2x_1 \frac{\partial}{\partial b} + x_2x_1{}^2 \frac{\partial}{\partial c} - x_1{}^3 \frac{\partial}{\partial d}\right)\Delta = 2J.$$

23. If f and J represent respectively the points α_i, α_j, α_k and β_i, β_j, β_k the polar of α_i as to f is α_i (§97) and a second point, *viz.*, β_i.

24. The pairs of points α_i, α_j; β_i, β_j and H belong to a quadratic involution whose double points are α_k, β_k.

25. The double (triple) points of the cubic involution $f + \lambda J = 0$ are given by the Jacobian of *any* two cubics of the pencil.

26. If $f = ax^3 + dy^3$, then $J = ad(x^3 - y^3)$, $H = adxy$, $\Delta = a^2d^2$. Denoting by $f_{\lambda\mu}$ the syzgetic pencil considered as a new form, thus $f_{\lambda\mu} \equiv \lambda f + \mu J \equiv a(\lambda + ad\mu)x^3 + d(\lambda - ad\mu)y^3$, obtain the complete system of $f_{\lambda\mu}$ in terms of the concomitants of f. *Ans.* $H_{\lambda\mu} = \theta H$, $\Delta_{\lambda\mu} = \theta^2\Delta$, $J_{\lambda\mu} = \theta(\lambda J + \mu\Delta f)$, where $\theta = \lambda^2 - \Delta\mu^2$. Hence all the forms of the syzygetic pencil, neglecting a numerical factor, have the same discriminant and the same Hessian. Such invariant forms are called *combinants* of the two base forms f and J.

27. The condition that f and J be apolar is $\Delta = 0$.

28. J can be written in the determinant form

$$J \equiv \begin{vmatrix} x_1^3 & 3x_1^2x_2 & 3x_1x_2^2 & x_2^3 \\ c & -2b & a & 0 \\ d & -c & -b & a \\ 0 & d & -2c & b \end{vmatrix}$$

116. The complete system of the binary quartic f consists of the two invariants I_2 and I_3 already noticed and, in addition to the quartic itself, two covariants,—namely the Hessian H, a $C_{2,4}$ and a *sextic covariant* J, a $C_{3,6}$ which is the Jacobian of f and H. For convenience of reference these five forms are written in full:

$$f : ax_1^4 + 4bx_1^3x_2 + 6cx_1^2x_2^2 + 4dx_1x_2^3 + ex_2^4$$

$$I_2 : ae - 4bd + 3c^2$$

$$I_3 : ace + 2bcd - ad^2 - b^2e - c^3$$

$$H : (ac - b^2)x_1^4 + 2(ad - bc)x_1^3x_2 + (ae + 2bd - 3c^2) x_1^2x_2^2 + 2(be - cd)x_1x_2^3 + (ce - d^2)x_2^4.$$

$$J : (a^2d - 3abc + 2b^3)x_1^6 + (a^2e + 2abd - 9ac^2 + 6b^2c)x_1^5x_2 + 5(abe - 3acd + 2b^2d)x_1^4x_2^2 - 10(ad^2 - b^2e)x_1^3x_2^3 - 5(ade - 3bce + 2bd^2)x_1^2x_2^4 - (ae^2 + 2bde - 9c^2e + 6cd^2)x_1x_2^5 - (be^2 + 2d^3 - 3cde)x_2^6.$$

The discriminant Δ is not included among the fundamental invariants since it is a rational, integral function of I_2 and I_3. For if f have a square factor it can be reduced to the form $ax_1^4 + 4bx_1^3x_2 + 6cx_1^2x_2^2$. The values of the invariants now are $I_2 = 3c^2$, $I_3 = -c^3$ from which we have at once the invariant relation $I_2^3 - 27I_3^2 = 0$. This invariant which vanishes when f has a double root must contain the discriminant as a factor. But since it is the same degree as the discriminant it must be the discriminant alone. Hence we may write

$$\Delta \equiv I_2^3 - 27I_3^2.$$

117. Interpretation of the invariants.—The invariant properties of the quartic, algebraic as well as geometric, are

most conveniently studied by means of the canonical form the complete system for which is:

$$\begin{aligned}
F &: x^4 + 6cx^2y^2 + y^4 \\
I_2 &: 1 + 3c^2 \\
I_3 &: c - c^3 \\
H &: cx^4 + (1 - 3c^2)x^2y^2 + cy^4 \\
J &: (1 - 9c^2)xy(x^4 - y^4)
\end{aligned}$$

to which may be added

$$\Delta : (1 - 9c^2)^2.$$

Since $F = 0$ is unaltered when x and y are interchanged or either is changed in sign, the roots $x/y = \alpha_1, \alpha_2, \alpha_3, \alpha_4$ may be taken respectively as

$$\alpha, \ -\alpha, \ 1/\alpha, \ -1/\alpha.$$

Hence, denoting by $\overline{ij}$ the difference $\alpha_i - \alpha_j$, the functions P, Q, R (§42) are

$$(\overline{12}\ \overline{34}) \equiv P = 4, (\overline{13}\ \overline{42}) \equiv Q = \left(\frac{\alpha^2 - 1}{\alpha}\right)^2, (\overline{14}\ \overline{23}) \equiv R = -\left(\frac{\alpha^2 + 1}{\alpha}\right)^2.$$

We have already noted both an algebraic and a geometric meaning of the vanishing of I_3: Algebraically it means that the quartic has an apolar quadratic and as a consequence can be written as the sum of two fourth powers; geometrically it means that the quartic represents four harmonic points.

Let us find the condition that the four points be equianharmonic, (§43 Ex. 14), *i. e.*, that one double ratio say $\rho = -R/P = -\omega$ where $-\omega = \frac{1}{2}(1 - i\sqrt{3})$. We ask that

$$\rho \equiv \frac{(\alpha^2 + 1)^2}{4\alpha^2} = -\omega.$$

We are thus led to the quartic

$$\alpha^4 + 2(1 + 2\omega)\alpha^2 + 1 = 0, \tag{1}$$

for which

$$I_2 = \tfrac{4}{3}(1 + \omega + \omega^2) = 0.$$

Conversely if $I_2 = 0$ one double ratio of the four points α is $-\omega$. Therefore

The necessary and sufficient condition that a binary quartic represent a set of four equi-anharmonic points is $I_2 = 0$.

From two ordinary invariants can always be derived an absolute invariant. Thus in the present instance when we apply to F a linear transformation of determinant D, I_2 is changed into $D^4 I_2$ and I_3 into $D^6 I_3$. Hence dividing $I_2{}^3$ by $I_3{}^2$ we obtain an invariant free of the modulus, *i. e.*,

The quartic has one absolute invariant, viz., $I_2{}^3/I_3{}^2$.

Any double ratio of the four points α is also an absolute, however an irrational, invariant of F. To express the absolute invariant of the quartic in terms of double ratios we first solve $F = 0$ for one value of α^2, thus

$$\alpha^2 = -3c + \sqrt{9c^2 - 1}. \tag{2}$$

Then denoting by r the double ratio $-R/P$ we have

$$r = -\frac{R}{P} = \frac{(\alpha^2 + 1)^2}{4\alpha^2} = \frac{(1 - 3c + \sqrt{9c^2 - 1})^2}{4(-3c + \sqrt{9c^2 - 1})}, \tag{3}$$

or rationalizing the denominator and simplifying

$$r = \frac{1 - 3c}{2}, \qquad c = \frac{1 - 2r}{3}. \tag{4}$$

Substituting this value of c in the canonical values of the invariants we find

$$I_2 = 1 + 3c^2 = \tfrac{4}{3}(r^2 - r + 1)$$
$$I_3 = c(1 - c^2) = \tfrac{4}{27}(r + 1)(r - 2)(2r - 1)$$

whence

$$\frac{I_2{}^3}{I_3{}^2} = 108\,\frac{(r^2 - r + 1)^3}{\{(r + 1)(r - 2)(2r - 1)\}^2}. \tag{5}$$

Now this equation is unaltered when r is replaced by each of the six double ratios (§42), consequently

The roots of the sextic equation (5) *are the six double ratios of the four points* f.

The relation (5) affords a pretty proof of the geometric implications of the three principal invariants. For if $I_3 = 0$, $r = -1$, 2, or $\frac{1}{2}$ and conversely, which proves the harmonic property (§43, Ex. 13). Likewise if $I_2 = 0$, $r^2 - r + 1 = 0$, *i. e.*, $r = -\omega$, $-\omega^2$ and conversely which proves the equi-anharmonic property.

Again if two of the points coincide, $r = 0$, 1 or ∞ (§43). In either case $I_2^3 - 27I_3^2 \equiv \Delta = 0$. On the other hand if $I_2^3 - 27I_3^2 = 0$, we find $r^2(r-1)^2 = 0$, *i. e.*, $r = 0$, 1 or ∞ and two of the points coincide. This establishes the geometric significance of the discriminant.

118. The sextic covariant.—A glance at the canonical form reveals an interesting property of J. For neglecting a constant, J is the product of the quadratics xy, $x^2 - y^2$, $x^2 + y^2$ which we shall denote respectively by q_1, q_2, q_3. It follows (§115, Ex. 7) that

1°. *The sextic covariant of a quartic breaks up into three mutually apolar quadratics.* Or geometrically, $J = 0$ *represents three pairs of mutually harmonic points.*

We shall now examine the relation of the sextic covariant to the quartic. From the definition of J as the Jacobian of the quartic and its Hessian we have (§102, 3°).

2°. $J = 0$ *gives the double points of the quartic involution* $f + \lambda H = 0$.

We shall return to this later. Again referring to the canonical form it appears that the linear factors of q_1 are precisely the x and y in the canonical equation of f. But it will be remembered that the quartic can be reduced to the canonical form in three ways and it follows from the symmetry characterizing the invariant theory that the

other quadratic factors of J provide the variables for the other two canonical forms. Or it can be verified directly that the polar of each of the quadratics q with respect to f is a constant multiple of the quadratic itself which proves the statement (§110, 1°, 2°). Hence

3°. *The three pairs of variables in the three canonical forms of the quartic are given by the quadratic factors q of the sextic covariant.*

Consider next the quadratic involutions connected with the quartic. The four points α_i (§117) can be paired in three ways as in the functions P, Q, R and each pairing determines a quadratic involution. We shall now prove that the double points of these involutions are given by the quadratics q. For the three groupings in pairs are

$$\alpha, \quad -\alpha \qquad\qquad 1/\alpha, \ -1/\alpha \tag{1}$$
$$\alpha, \quad 1/\alpha \qquad\qquad -1/\alpha, \ -\alpha \tag{2}$$
$$\alpha, \ -1/\alpha \qquad\qquad -\alpha, \quad 1/\alpha \tag{3}$$

Thence, substituting in equation (2) §86 and setting $x = x'$, the double points are found to be $xy = 0$, $x^2 - y^2 = 0$, $x^2 + y^2 = 0$. Combining this result with 1° we may say

4°. *The four points f set up three quadratic involutions along the line each of which has for double points one pair q_i of J and in which q_j and q_k represent conjugate pairs.*

119. The syzygy connecting the fundamental forms of the quartic[1] can be derived by an application of the theorems in the preceding section. For from the values of the invariants I_2, I_3 (§117) we have the relation

$$4c^3 - I_2 c + I_3 = 0^{[2]} \tag{1}$$

[1] There must be a syzygy since only four of these forms can be independent (§107).

[2] This equation is another form of the resolvent cubic of the quartic (§110). In fact writing $c = -k/2$ the equation becomes $k^3 - I_2 k - 2I_3 = 0$ which is the earlier form.

and combining f and H, the equation

$$H - cf = (1 - 9c^2)x^2y^2 \tag{2}$$

where c must satisfy (1). Now the right side of (2) is, but for a constant factor, the square of q_1. But since there are three values of c there are two other relations like (2) giving the squares of the other two quadratic factors of J. Therefore denoting the values of c by c_1, c_2, c_3, we must have

$$4(H - c_1f)(H - c_2f)(H - c_3f) \equiv k(q_1q_2q_3)^2 \equiv k\,J^2 \tag{3}$$

where k is a constant. Observe that the left side of (3) set equal to zero is the same equation in H/f that (1) is in c since their roots are identical. Hence (3) may be written in the form

$$4H^3 - I_2Hf^2 + I_3f^3 \equiv kJ^2$$

wherein, substituting values of the several forms, k is found to be -1 and the required syzygy is

$$-J^2 \equiv 4H^3 - I_2Hf^2 + I_3f^3. \tag{4}$$

We should have arrived at the same result of course had we sought to express J^2 directly by means of the complete system. For J^2 is a $C_{6,12}$ and the only combinations of the fundamental forms of degree 6 and order 12 are H^3, I_2Hf^2 and I_3f^3. Hence there must be a relation like

$$J^2 \equiv k_1H^3 + k_2I_2Hf^2 + k_3I_3f^3$$

where the k's are fixed by identifying coefficients.

120. Syzygetic pencil of binary quartics.—Since the quartic and its Hessian are of the same order there is associated with the two forms the syzygetic pencil of quartics $\lambda f + \mu H$. Or geometrically, a quartic and its Hessian define the quartic involution

$$\lambda f + \mu H = 0, \tag{1}$$

comprising a singly infinite system of *tetrads* of points.

Now the canonical form of the Hessian is the same as that of the quartic, hence the canonical equation of either,—

with c as parameter,—may be regarded as defining the involution. Or utilizing the results of §118 we may say

The involution (1) *can be written in three ways in the canonical form*

$$F_{\lambda\mu} \equiv \lambda(x^4 + y^4) + 6\mu x^2y^2 = 0$$

if we take for new variables the linear factors of the quadratics.

Let us examine the involution for special sets. First I_2 calculated for F involves the parameter λ/μ to the second degree, hence

1°. *The involution contains two equi-anharmonic sets corresponding to* $I_2 = 0$.

Likewise I_3 contains λ/μ to the third degree, therefore

2°. *There are three values of* λ/μ *given by* $I_3 = 0$ *for which the syzygetic pencil represents four harmonic points.*

There are however in general six sets in the involution with a specified double ratio, since equation (5) §117 formed for F is for given r of degree six in λ/μ.

The discriminant $\lambda^2(\lambda^2 - 9\mu^2)^2$ of $F_{\lambda\mu}$ which is of the sixth degree in λ/μ is however the square of a cubic. Accordingly the six tetrads having a double point of the involution coalesce into three. Corresponding to each root of $\Delta_{\lambda\mu} = 0$ we find in fact the square of one of the quadratics q. Hence if two points of a set coincide the other two do also and there is a *pair* of double points. Summarizing

3°. *The involution contains three and only three sets given by* $\Delta_{\lambda\mu} = 0$ *which have a double point, namely the sets* q_i^2 *which have two double points each.*

Except for a constant factor the covariant J is the same for every quartic of the syzygetic pencil.[1] Consequently

Every tetrad of the involution can be broken up into three groups of two pairs each, the two pairs in each group belonging to one of the quadratic involutions associated with f.

[1] That is J is a combinant of the pencil, §115, Ex. 26.

121. Steinerians.—We have had two geometric interpretations of the Hessian of a binary form:

(*a*) The locus of a point x whose $(n - 2)$th (quadratic) polar set is a repeated point y

(*b*) The locus of double points x in the first polar system of y

$$y_1\frac{\partial f}{\partial x_1} + y_2\frac{\partial f}{\partial x_2} \tag{1}$$

where y_1/y_2 is a parameter. Or combining the two

If the first polar of y have a double point x, then the $(n - 2)$th polar of x has a double point y, the condition in either case being $H(x) = 0$.

The locus of points y associated with the Hessian points x in this reciprocity we shall call the *Steinerian*[1] S of f. Formally

The Steinerian is at once (a) the locus of double points of quadratic polars and (b) the locus of points whose first polars have double points.

As the Hessian $H(x)$ is the discriminant as to y of the $(n - 2)$th polar of x, so *the Steinerian $S(y)$ is the discriminant as to x of the first polar* (1) *of y.* It follows that the Steinerian of a binary form (but of a binary form only) has the same order[2] as the Hessian which is obvious geometrically, the correspondence between the two sets of points being (1, 1).

It is evident from the definition that the Steinerian points are projectively attached to f, hence S is a covariant of f and must be expressible in terms of the complete system. To find the Steinerian of the quartic in canonical form we write the cubic involution of first polars

$$\left(x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y}\right)F \equiv x'x^3 + 3cy'x^2y + 3cx'xy^2 + y'y^3.$$

[1] After the great German geometer Steiner.

[2] The order of the Steinerian of a ternary form is $3(n - 2)^2$.

The discriminant of this cubic is the Steinerian, hence

$$S \equiv 4c^3 (x'^4 + y'^4) + (1 - 6c^2 - 3c^4) x'^2y'^2.$$

Since the Steinerian is a $C_{4,4}$ it must be a linear combination of I_2H and I_3f, *i. e.*, $S \equiv k_1I_2H + k_2I_3f$. Equating coefficients in these two identical expressions for S we find $k_1 = 1$, $k_2 = -1$, or

$$S = I_2H - I_3f.$$

Hence *the Steinerian points of the quartic are a special set in the syzygetic pencil.*

122. Higher binary forms.—We have designated the forms here considered as elementary because of their relative simplicity. There is a sharp distinction between the quartic and lower forms on the one hand and the quintic and higher forms on the other so that the methods which suffice in the one case are not adequate in the other. There is no difficulty in constructing a table of concomitants of a system of forms (§107), the difficulty lies in obtaining a *complete* system which shall contain the minimum number of forms. The work of deriving complete systems increases enormously with the order both because of the number of forms involved and the complexity of individual forms. Thus the complete system for the quintic, first obtained by Gordan with the symbolic method, consists of 23 forms, including four invariants one of which is of the 18th degree. The system of the sextic contains 26 forms five of which are invariants. The vanishing of one of these invariants is the condition that the sextic represent three pairs of points in a quadratic involution which sounds like an innocent requirement but the invariant is of of the 15th degree and written at length occupied thirteen pages in the second edition of Salmon's *Higher Algebra*. Von Gall has published complete systems for the septimic and the octavic each of which contains over a hundred forms.

Several simultaneous systems have also been obtained, *e. g.*, that for a cubic and quadratic, two cubics, quadratic and quartic, and cubic and quartic.

All the systems mentioned in this section are discussed in the books of Salmon and Clebsch, except the results of von Gall which are printed in the *Mathematische Annalen*. A very full bibliography (*Bericht über den gegenwärtigen Stand der Invariantentheorie*) has been compiled by W. Franz Meyer and published in the *Jahresbericht der Deutschen Mathematiker Vereinigung*, Vol. I, 1892.

EXERCISES

1. The Hessian and sextic covariant of the quartic can be written in the determinant forms:

$$H \equiv - \begin{vmatrix} 0 & x_2^2 & -x_2x_1 & x_1^2 \\ x_2^2 & a & b & c \\ -x_1x_2 & b & c & d \\ x_1^2 & c & d & e \end{vmatrix}.$$

$$J \equiv \frac{1}{3} \begin{vmatrix} x_1^3 & 3x_1^2x_2 & 3x_1x_2^2 & x_2^3 & 0 \\ 0 & x_1^3 & 3x_1^2x_2 & 3x_1x_2^2 & x_2^3 \\ -d & 3c & -3b & a & 0 \\ -e & 2d & 0 & -2b & a \\ 0 & -e & 3d & -3c & b \end{vmatrix}.$$

2. Show that the Hessian of a quartic can be obtained by operating on I_3 as follows:

$$\left(x_1^4 \frac{\partial}{\partial e} - x_1^3x_2 \frac{\partial}{\partial d} + x_1^2x_2^2 \frac{\partial}{\partial c} - x_1x_2^3 \frac{\partial}{\partial b} + x_2^4 \frac{\partial}{\partial a}\right) I_3 = H.$$

What is the result of applying this operation to I_2?

3. The discriminant of the resolvent cubic $k^3 - I_2k - 2I_3 = 0$ of the quartic is the same as that of the quartic.

4. Solve this cubic for the canonical form.

5. The invariants I_2 and I_3 of $F_{\lambda\mu} \equiv \lambda f + \mu H$ are numerical multiples respectively of the Hessian and cubicovariant of the cubic $\lambda^3 - I_2\lambda\mu^2 - 2I_3\mu^3 = 0$ obtained from the resolvent cubic by writing $k = \lambda/\mu$.

6. If a quartic have a double factor, H has the same double factor and J has it for a five-fold factor.

7. The necessary and sufficient condition that a quartic have a triple factor is $I_2 = 0$, $I_3 = 0$.

8. The necessary and sufficient condition that a quartic be a perfect fourth power is $H \equiv 0$.

9. If a quartic have two square factors it is a multiple of its Hessian (Ex. 9, §111), *i. e.*,

$$\frac{ac - b^2}{a} = \frac{ad - bc}{2b} = \frac{ae + 2bd - 3c^2}{6c} = \frac{be - cd}{2d} = \frac{ce - d^2}{e},$$

and conversely.

10. Find the invariant relation between f and H in Ex. 9. (The quartic may be taken as $6cx^2y^2$.) *Ans.* $3I_3f - 2I_2H = 0$.

11. The necessary and sufficient condition that a quartic have two square factors is $J \equiv 0$. The conditions are the same as in Ex. 9.

12. The Hessian of the Hessian of a quartic belongs to the syzygetic pencil. Find an invariant expression for it. *Ans.* $3I_3f - I_2H$.

13. If a quartic have a triple factor H has the same factor quadruply and J sextuply. If H is a perfect fourth power, f has a triple factor. Apply 7, 8 and 12.

14. Show that the invariants I_2 and I_3 of the composite quartic $f \equiv (ux + vy)(ax^3 + 3bx^2y + 3cxy^2 + dy^3)$ are the two covariants of the cubic component when u and v are replaced by y and $-x$; that the discriminant of f is the discriminant of the cubic multiplied by the square of the cubic. Thence from the expression for the discriminant of the quartic in terms of the fundamental invariants find the syzygy among the forms of the cubic.

15. Both the quartic and its Hessian are apolar to the sextic covariant for the polar of either with respect to J would be a quadratic covariant (theorem of Boole) but the complete system of f contains no covariant of order less than 4. The two polars must therefore vanish identically. The student may verify with the canonical forms.

16. Show that the equation connecting the double ratios of four points with the absolute invariant of the quartic ((5) §117) can be written in the form

$$\frac{27I_2^3}{4(I_2^3 - 27I_3^2)} = \frac{(r^2 - r + 1)^3}{r^2(r - 1)^2}.$$

Write this equation as a cubic in $r(1 - r)$, showing that the original sextic can be solved by solving a cubic.

17. Express each of the double ratios of the four points represented by a quartic f in terms of the invariants (I_2, I_3, Δ) verifying thus that they are irrational invariants.

18. Express in terms of the fundamental forms of the quartic the following invariants and covariants:

(*a*) Condition for f and H to be apolar
(*b*) Condition for H to be self-apolar
(*c*) Condition for J to be self-apolar
(*d*) Jacobian of f and J
(*e*) Jacobian of H and J
(*f*) Hessian of J.

Answers, neglecting numerical factors: (*a*) I_3, (*b*) I_2^2, (*c*) Δ, (*d*) $I_2f^2 - 12H^2$, (*e*) $f(3I_3f - 2I_2H)$, (*f*) $I_2^2f^2 - 36I_3Hf + 12I_2H^2$.

19. Every member of the syzygetic pencil of the quartic is apolar to all second polars of J.

20. The Steinerian of a binary cubic is identical with the Hessian. (This theorem holds for the ternary cubic as well.)

21. Follow the method of §110 to find those cubics which are, to within a constant multiple, their own polars with respect to a binary sextic. Write for the sextic (in determinant form) the k-equation corresponding to the resolvent cubic of the quartic. Write a similar equation for the octavic, for the binary $2n$-ic. The constant term in each equation is the catalecticant of the form in question.

22. If the binary quartic is written in the canonical form $ax^4 + by^4 + cz^4$, where $x + y + z = 0$, find the values of the invariants I_2, I_3 and Δ. *Ans.* $I_2 = bc + ca + ab$, $I_3 = abc$, $\Delta = (bc)^{1/3} + (ca)^{1/3} + (ab)^{1/3}$.

23. Show that the value of Δ in Ex. 22 is $I_2^3 - 27I_3^2$.

24. If two quartics are each apolar to a third, the Jacobian of the two is apolar to the sextic covariant of the third.

CHAPTER X

ANALYTIC TREATMENT OF THE CONIC

PART I. THE CONIC AS A RATIONAL CURVE

123. Parametric equations of the conic.—Consider the projective coördinates x_1, x_2, x_3 of a point as defined by three linearly independent binary quadratics thus

$$\begin{aligned} x_1 &= a_1t^2 + 2b_1t + c_1 \\ x_2 &= a_2t^2 + 2b_2t + c_2 \\ x_3 &= a_3t^2 + 2b_3t + c_3 \end{aligned} \tag{1}$$

where the coefficients are regarded as fixed constants and where t is a parameter. Then every value of t determines a unique set of value of x_1, x_2, x_3 and therefore a definite point x in the plane.[1] And as t assumes all values (ranges along a line) the point x will trace out a curve in the plane. Equations (1) are called *parametric equations of the curve in points* and the value of t corresponding to a point is the *parameter of the point.*

A curve like (1) which is the locus of points whose ternary coördinates can be expressed as rational integral functions of a single parameter is defined to be a *rational (point) curve.*

To find the order of the curve we need only ascertain the number of points in which it meets an arbitrary line. Now the parameters of the points in which a line u

$$(ux) \equiv u_1x_1 + u_2x_2 + u_3x_3 = 0 \tag{2}$$

[1] It is convenient to designate the point (x_1, x_2, x_3) as the point x.

cuts the curve are found by substituting in (2) the values of the x's from (1), the result of which is

$$(au)t^2 + 2(bu)t + (cu) = 0 \tag{3}$$

a quadratic in t. Accordingly the line has two points in common with the curve which is therefore a conic.

We should be led to the same results if instead of eliminating λ between the equations of the two projective pencils (§63)

$$\begin{aligned}(\alpha x) + \lambda(\beta x) &\equiv (\alpha_1 + \lambda\beta_1)x_1 + (\alpha_2 + \lambda\beta_2)x_2 \\ &\qquad + (\alpha_3 + \lambda\beta_3)x_3 = 0 \\ (\gamma x) + \lambda(\delta x) &\equiv (\gamma_1 + \lambda\delta_1)x_1 + (\gamma_2 + \lambda\delta_2)x_2 \\ &\qquad + (\gamma_3 + \lambda\delta_3)x_3 = 0\end{aligned}$$

we should combine both with the line $(ux) = 0$. Eliminating x from these three equations we obtain the determinant

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ \alpha_1 + \lambda\beta_1 & \alpha_2 + \lambda\beta_2 & \alpha_3 + \lambda\beta_3 \\ \gamma_1 + \lambda\delta_1 & \gamma_2 + \lambda\delta_2 & \gamma_3 + \lambda\delta_3 \end{vmatrix} \equiv u_1q_1(\lambda) + u_2q_2(\lambda) + u_3q_3(\lambda) = 0 \tag{4}$$

where the q's are quadratics in the parameter λ.

For given λ, (4) is linear in u and hence the equation of a point,—the point of intersection of corresponding lines of the two pencils. In other words it is a point of the conic. While for given u it is quadratic in λ and gives the two parameters on the conic cut out by the line u. Since (§64) through the mediation of λ, a (1, 1) correspondence is established between the points of the conic and a line, equation (4) is said to *map* the line onto the conic and is called in consequence a *map equation of the conic.*

Since the coördinates of a point are merely the coefficients in its equation we have from (4)

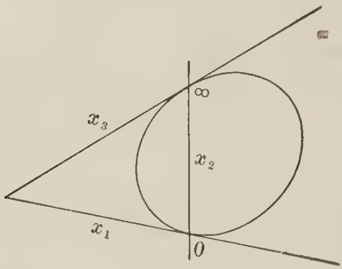
$$x_1 = q_1, \qquad x_2 = q_2, \qquad x_3 = q_3 \tag{5}$$

which are parametric equations of the conic in terms of the parameter λ.

From the map equation (3) *of the conic in points can be derived at once the ternary line equation.* For a tangent line cuts out two coincident parameters and the required condition is that the discriminant of (3) considered as a quadratic in t should vanish. That is the line equation of conic (1) or (3) is

$$(au)(cu) - (bu)^2 = 0. \tag{6}$$

124. Canonical form of parametric point equations.—We proceed to show how the general parametric equations of the conic (1) of the previous section can be simplified. There the sides of the triangle of reference cut the conic in points whose parameters are given by the general quadratics $a_i t^2 + 2b_i t + c_i = 0$. But any three lines (not on a point) may be taken as reference triangle. In particular



we may choose the tangents at the points with parameters $t = 0$, $t = \infty$ and the chord joining these points. This requires that we equate one x to a quadratic with both roots zero, a second to a quadratic with both roots infinite and the third to a quadratic with roots $0, \infty$. The parametric equations of the conic then assume the *canonical form*

$$x_1 = t^2, \qquad x_2 = 2t, \qquad x_3 = 1, \tag{1}$$

and the map equation the canonical form

$$u_1 t^2 + 2u_2 t + u_3 = 0. \tag{2}$$

Thence the *ternary line equation* is found at once to be

$$u_2^2 - u_3u_1 = 0. \tag{3}$$

We can also write down by inspection the *ternary point equation*, observing that this equation must be (*a*) of the second order, (*b*) homogeneous in the x's and (*c*) free of t. Such a relation is obviously

$$x_2^2 - 4x_3x_1 = 0. \tag{4}$$

125. Parametric equations of the conic in lines.—Dually if the projective coördinates of a line u are defined by three linearly independent binary quadratics in a parameter t

$$u_i = a_it^2 + 2b_it + c_i, \qquad i = 1, 2, 3, \tag{1}$$

the locus of u is a rational curve of the second class (conic) of which (1) are the parametric equations.

The parameters of the two lines common to the conic and an arbitrary point

$$(ux) \equiv u_1x_1 + u_2x_2 + u_3x_3 = 0, \tag{2}$$

found by combining (1) and (2) are given by

$$(ax)t^2 + 2(bx)t + (cx) = 0 \tag{3}$$

considered as a quadratic in t. Equally for given t (3) is the equation of a line of the conic, namely the line with parameter t. Equation (3) is called the *map equation of the conic in lines* since it maps the lines of a pencil into the lines of the conic in a one-to-one way.

The *ternary point equation*, which is the condition that the two lines on the point have coincident parameters, is the discriminant as to t of (3) or

$$(ax)(cx) - (bx)^2 = 0. \tag{4}$$

By choosing for reference 3-point the contacts of lines

with parameters 0 and ∞ and their intersection the conic can be written in the *canonical form*

$$u_1 = t^2, \quad u_2 = 2t, \quad u_3 = 1. \tag{5}$$

The corresponding map equation is

$$x_1t^2 + 2x_2t + x_3 = 0, \tag{6}$$

and the ternary point equation

$$x_2^2 - x_3x_1 = 0. \tag{7}$$

Finally we have by inspection the ternary line equation

$$u_2^2 - 4u_3u_1 = 0. \tag{8}$$

126. Line joining two points.—If the conic is taken in the canonical form

$$x_1 = t^2, \quad x_2 = 2t, \quad x_3 = 1, \tag{1}$$

the ternary coördinates of the points with parameters t_1 and t_2 are $(t_1^2, 2t_1, 1)$ and $(t_2^2, 2t_2, 1)$ and the line joining the two is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ t_1^2 & 2t_1 & 1 \\ t_2^2 & 2t_2 & 1 \end{vmatrix} = 0. \tag{2}$$

The determinant evidently vanishes when $t_1 = t_2$, hence $t_1 - t_2$ is a factor. Removing this factor and expanding we have as *the line* $\overline{t_1t_2}$

$$2x_1 - (t_1 + t_2)x_2 + 2t_1t_2x_3 = 0. \tag{3}$$

When $t_1 = t_2 = t$ (3) reduces to

$$x_1 - tx_2 + t^2x_3 = 0 \tag{4}$$

which is therefore the *equation of the tangent at the point t.*

The coördinates of the tangent line and, for variable t, the *parametric equations of the conic in lines* are manifestly[1]

$$u_1 = 1, \quad u_2 = -t, \quad u_3 = t^2. \tag{5}$$

[1] The student will not fail to note that we employ the same parameter to name both the points and lines of the conic, that in fact *the point t refers to the contact of the line t.* When there is occasion to make a distinction T may be used for the line parameter.

We have thus incidentally solved the important problem of passing from the parametric point equations to parametric line equations, or by duality the reverse. And by utilizing earlier results we can derive both ternary equations as well, as indicated in the diagram.

$$\begin{tikzcd}
\text{param. pt. eqs.} \arrow[r, leftrightarrow] \arrow[dr] & \text{param. line eqs.} \arrow[dl] \\
\text{ternary pt. eq.} & \text{ternary line eq.}
\end{tikzcd}$$

127. The ternary point equation can however be written down directly from the parametric point equations as a consequence of Ex. 11, §115 thus

$$\begin{vmatrix} D_{11} & D_{12} & D_{13} & x_1 \\ D_{21} & D_{22} & D_{23} & x_2 \\ D_{31} & D_{32} & D_{33} & x_3 \\ x_1 & x_2 & x_3 & 0 \end{vmatrix} = 0. \tag{1}$$

A second way to solve the problem is to eliminate t according to Sylvester's dialytic method.[1] For this purpose we introduce a factor of proportionality ρ and write the conic

$$\rho x_i = a_i t^2 + b_i t + c_i, \qquad i = 1, 2, 3.$$

Then multiplying each equation by t we obtain the three new equations

$$t\rho x_i = a_i t^3 + b_i t^2 + c_i t.$$

Eliminating now the six quantities $t\rho$, ρ, t^3, t^2, t, 1 from the six equations we get the ternary equation in the determinant form

$$\begin{vmatrix} x_1 & 0 & a_1 & b_1 & c_1 & 0 \\ x_2 & 0 & a_2 & b_2 & c_2 & 0 \\ x_3 & 0 & a_3 & b_3 & c_3 & 0 \\ 0 & x_1 & 0 & a_1 & b_1 & c_1 \\ 0 & x_2 & 0 & a_2 & b_2 & c_2 \\ 0 & x_3 & 0 & a_3 & b_3 & c_3 \end{vmatrix} = 0. \tag{2}$$

[1] See for example Burnside and Panton, *Theory of Equations*, Vol. II, §154.

This method may be used to find the ternary equation of a rational curve of order n but it gives rise to a determinant of order $3n$.

Richmond's method[1] is equally general and leads to a determinant of order n. Suppose the parametric equations of the rational curve are

$$x_1 = f_1(t), \; x_2 = f_2(t), \; x_3 = f_3(t), \tag{3}$$

where f_i are binary forms of order n. Then if x_1, x_2, x_3 are coördinates of the point t while s is the parameter of an arbitrary point of the curve, the determinant

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ f_1(t) & f_2(t) & f_3(t) \\ f_1(s) & f_2(s) & f_3(s) \end{vmatrix} \equiv 0 \tag{4}$$

for every s, since by (3) $x_1 : x_2 : x_3 : = f_1(t) : f_2(t) : f_3(t)$. Removing the obvious factor $t - s$ from (4) and expanding we obtain a function $F(s)$, of degree $n - 1$ in s, which must vanish identically. Thence equating to zero the coefficients of $1, s, \ldots s^{n-1}$ in F, we get n equations of degree $n - 1$ in t. Finally eliminating $1, t, \ldots t^{n-1}$ from these n equations, the ternary equation of the curve appears as a determinant of order n.

For example, equation (3) §126 may be written

$$F(t_2) \equiv (2x_3t_1 - x_2)t_2 - x_2t_1 + 2x_1.$$

If $F \equiv 0$, we must have

$$2x_3t_1 - x_2 = 0, \quad -x_2t_1 + 2x_1 = 0.$$

Eliminating t_1, the ternary equation of the conic is

$$\begin{vmatrix} 2x_3 & -x_2 \\ -x_2 & 2x_1 \end{vmatrix} = 0, \text{ or } 4x_3x_1 - x_2^2 = 0.$$

[1] *Bulletin*, Amer. Math. Soc., Nov., 1916, p. 90.

EXERCISES

1. If the conic is written parametrically $x_1 = t^2$, $x_2 = (t-1)^2$, $x_3 = 1$, show that the reference triangle circumscribes the conic, Derive the parametric lines equations and both ternary equations.

2. Write the parametric equations in points and lines and the ternary equations when the triangle of reference is inscribed, with vertices at $t = 0, 1, \infty$.

3. Expand the two determinants of §127 and thus identify these two forms of the equation of a conic. (Replace b_i in (2) by $2b_i$.)

4. Prove that the conic of §123 is general. (While each quadratic appears to contain two essential constants, the equation of the conic (1), §127, reduces the number by one. They cannot be reduced further since (1) is the *locus* of x. The number of essential constants is therefore precisely 5.) Hence a conic is always rational.

5. Prove from the parametric representation that a curve of order two is also of class two.

6. Given the line conic $u_1 = t^2 + 1$, $u_2 = t^2 - 1$, $u_3 = 2t$, find the parametric point equations and the two ternary equations.

7. Find the ternary line equation of the conic in Ex. 6 by using the dual of equation (1) §127.

8. Show that if the three quadratics (1) §123 have a common factor the ternary line equation of the conic is a square. What happens to the conic? Are the three quadratics linearly independent?

In the remaining exercises the conic should be taken in the canonical form $x_1 = t^2$, $x_2 = 2t$, $x_3 = 1$, and the results of §§124-126 collected for reference.

9. The unit line (line with coördinates (1, 1, 1)) is tangent to the conic at $t = -1$. Is the unit point on the conic?

10. Find the ternary coördinates of the point of intersection of the lines with parameters t_1 and t_2. *Ans.* The three quadratics x_i polarized with respect to t_1 and t_2.

11. Find the equation of the line cutting out the two parameters given by the quadratic $\alpha t^2 + 2\beta t + \gamma = 0$.

12. Find the equation of the tangents to the conic from (y_1, y_2, y_3). Find the polar line of this point.

13. Show that the two quadratics giving the parameters of (a) points common to a line u and the conic and (b) lines common to a point x and the conic will be apolar if $(ux) = 0$, *i. e.*, if the point x and line u are incident.

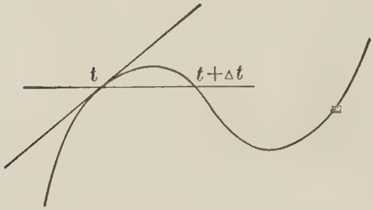
14. Prove that a triangle and its polar triangle (Ex. 11, §74) are per-

spective. (Take as the pairs of points in which sides of one of the triangles cuts the conic $\alpha_i t^2 + 2\beta_i t + \gamma_i$ and use Exs. 10, 11.)

15. Obtain the parametric line equations of a conic when the conic is defined as the locus of junctions of corresponding points of two projective ranges.

16. Let $x_i = f_i$, and $u_i = \phi_i$, $i = 1, 2, 3$ be respectively the parametric point and line equations of a rational curve. What is represented geometrically by the roots of $f_i = 0$? $\phi_i = 0$?

128. Line equations of rational curve derived from point equations. First method.—The manner (§126) of finding the tangent line and thence the line equations of a conic written parametrically applies to any rational curve. The several steps involved are however really equivalent to a differentiation process which for the sake of practical simplicity we shall restate in the notation of the calculus.



Let the rational curve in points be written parametrically

$$x_1 = f_1(t), \qquad x_2 = f_2(t), \qquad x_3 = f_3(t),$$

where f_i are (non-homogeneous) binary forms of order n. Required the parametric line equations.

The ternary coördinates of a point t are $f_1(t)$, $f_2(t)$, $f_3(t)$ and those of a neighboring point $f_1(t + \Delta t)$, $f_2(t + \Delta t)$, $f_3(t + \Delta t)$. The junction of the two is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ f_1(t) & f_2(t) & f_3(t) \\ f_1(t+\Delta t), & f_2(t+\Delta t), & f_3(t+\Delta t) \end{vmatrix} = 0. \tag{2}$$

Now subtracting the second row from the third and dividing by Δt we have

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ f_1(t) & f_2(t) & f_3(t) \\ \dfrac{f_1(t+\Delta t) - f_1(t)}{\Delta t}, & \dfrac{f_2(t+\Delta t) - f_2(t)}{\Delta t}, & \dfrac{f_3(t+\Delta t) - f_3(t)}{\Delta t} \end{vmatrix} = 0. \tag{3}$$

Hence passing to the limit as $\Delta t \to 0$, we obtain as the *equation of the tangent at* t

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ f_1(t) & f_2(t) & f_3(t) \\ \dfrac{df_1}{dt} & \dfrac{df_2}{dt} & \dfrac{df_3}{dt} \end{vmatrix} = 0. \tag{4}$$

And the *parametric line equations* are from (4)

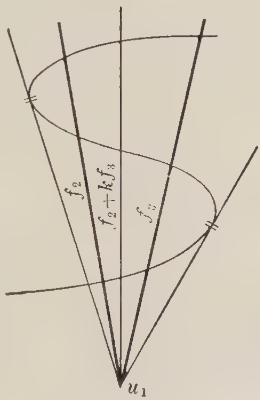
$$u_i = \text{coefficient of } x_i. \tag{5}$$

The second method leads to still simpler formulas for the line equations and is therefore perferable in the general case. We now replace t by a homogeneous parameter t_1/t_2 and write the equations of the curve

$$x_i \equiv f_i(t_1, t_2) = a_i t_1^n + b_i t_1^{n-1} t_2 + \quad . \quad . \quad . + r_i t_2^n, \qquad i = 1, 2, 3. \tag{6}$$

Then the pencil of lines $x_2 + kx_3 = 0$ will cut the curve in points whose parameters belong to the involution

$$f_2 + kf_3 = 0. \tag{7}$$



It is geometrically obvious that a line of the pencil which is tangent to the curve will cut out a set of the involution containing a double point. Now the double points of the involution, *i. e.*, the contacts of tangents from the vertex u_1 are given by the Jacobian J_{23} of f_2 and f_3 (§102, 3°). But the parameters J of the contacts are the same as the parameters of the tangent lines themselves (footnote, §126) so that $J = 0$ gives the parameters of the lines of the curve which are on the vertex

u_1 of the triangle of reference. It follows that *the line equations* are

$$u_1 = J_{23}, \qquad u_2 = J_{31}, \qquad u_3 = J_{12}. \tag{8}$$

Or if we write the determinant equation

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ \dfrac{\partial f_1}{\partial t_1} & \dfrac{\partial f_2}{\partial t_1} & \dfrac{\partial f_3}{\partial t_1} \\ \dfrac{\partial f_1}{\partial t_2} & \dfrac{\partial f_2}{\partial t_2} & \dfrac{\partial f_3}{\partial t_2} \end{vmatrix} = 0. \tag{9}$$

it is plain that the u_i of the line equations are the cofactors of x_i in the determinant while (9) is at once the map equation of the curve in lines and the equation of the tangent at the point t_1, t_2.

Example. In applying the method it is not necessary to make the equations actually homogeneous in the parameter. At any rate we need not change t, for in passing to the homogeneous forms we would set $t = t_1$ and then to get back to the non-homogeneous forms set $t_1 = t$. Thus we may write for the second row of (9) df_i/dt. Then to get the third row we may imagine the proper power of t_2 inserted to bring every term in the f's to the nth degree, form the partial derivative as to t_2 and then substitute $t_2 = 1$.

To illustrate let us calculate the line equations of

$$x_1 = t, \qquad x_2 = t^3, \qquad x_3 = t^4 + 1.$$

We have

$$\frac{dx_1}{dt} = 1, \qquad \frac{dx_2}{dt} = 3t^2, \qquad \frac{dx_3}{dt} = 4t^3$$

$$\left(\frac{\partial}{\partial t_2}\, t(t_2{}^3)\right)_{t_2 = 1} = 3t, \quad \left(\frac{\partial}{\partial t_2}\, t^3(t_2)\right)_{t_2 = 1} = t^3,$$

$$\left(\frac{\partial}{\partial t_2}\, \overline{t^4 + (t_2{}^4)}\right)_{t_2 = 1} = 4,$$

where we have written in parentheses the power of t_2 to be supplied mentally.

Hence the map equation in lines is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ 1 & 3t^2 & 4t^3 \\ 3t & t^3 & 4 \end{vmatrix} = 0$$

or

$$(t^6 - 3t^2)x_1 + (1 - 3t^4)x_2 + 2t^3x_3 = 0,$$

and the parametric line equations are

$$u_1 = t^6 - 3t^2, \qquad u_2 = -3t^4 + 1, \qquad u_3 = 2t^3.$$

129. Geometry on a rational curve.—It appears from the foregoing that the point is playing a double rôle in the theory of the conic (or any rational curve) according as it is considered an element (x_1, x_2, x_3) in the ternary or an element t in the binary domain. And it is essential that the distinction between the two be clearly recognized. We might maintain the distinction by speaking of a "point of the plane" or a "point with the ternary coördinates x_1, x_2, x_3" on the one hand and *a* "point of the conic" or a "point with the parameter t" on the other. Perhaps however we can, without danger of confusion, refer to the one simply as "the point x" and the other as "the point t." Dually the "line u" or the "line t" will have a similar significance.

Since an equation $f(t) = 0$ of order n determines n values of t we may say that *the binary form f represents n points t on the conic.* And the invariant properties of f can be translated into a projective *geometry on the conic* which is an exact analogue of the geometry on a line as presented in Chapters V, VII, IX. In the same way we get a geometry on any rational curve whether in the plane or a space of higher dimensions. We have thus a remarkable occurrence of an *infinite number of geometries with a common algebra,* namely

the invariant algebra of binary forms. Or to put it differently, the single abstract theory of invariant algebra has an infinite variety of concrete geometric representations.[1] The whole of these (with the duals) constitute the binary domain.

We shall now sketch the elements of the geometry on a conic, using the language of binary forms to characterize the geometric properties. Thus by four harmonic points t we mean four points whose parameters satisfy the equation $(t_1t_2|\, t_3t_4) = -1$. And by four self-apolar points we mean points whose parameters are roots of a self-apolar quartic. Similarly any projective property of a set of points t will be a property corresponding to some invariant relation on the parameter.

130. Quadratic involutions on the conic.—It is evident that *there must be a* (1, 1) *correspondence between lines in the plane and binary quadratics.* For a quadratic represents two points on a conic which in turn determine a line, and reciprocally. If the conic be in the canonical form

$$x_1 = t^2, \quad x_2 = 2t, \quad x_3 = 1, \tag{1}$$

and

$$u_1x_1 + u_2x_2 + u_3x_3 = 0 \tag{2}$$

is the equation of any line, then the corresponding quadratic is

$$u_1t^2 + 2u_2t + u_3 = 0, \tag{3}$$

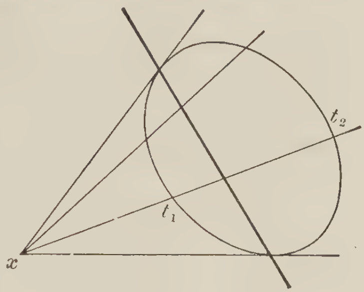
for (3) gives the points in which the line cuts the conic.

If the roots of (3) are t_1 and t_2, we have $t_1 + t_2 = -2u_2/u_1$ and $t_1t_2 = u_3/u_1$. Substituting these values of the u's in (2) we get the equation of the line corresponding to the quadratic in the form

$$2x_1 - (t_1 + t_2)x_2 + 2t_1t_2x_3 = 0. \tag{4}$$

[1] In fact there are infinitely many geometries in the plane alone, one for the rational curve of each order 1, 2, . . . n. Similarly there is an infinite number in space and also an infinite number in S_4 and so on.

Equation (4) which is the condition that the points t_1, t_2 of the conic be collinear with the point x of the plane is subject to a double interpretation. For given t_1, t_2 it is the equation of their junction (§126). Equally for constant x and variable t it defines the relation existing between pairs of variable points on the conic when the junctions of the pairs are on a fixed point x of the plane. But on the face of it equation (4) represents a quadratic involution of which t_1 and t_2 are conjugate points. That is



The lines of a pencil on any point x in the plane cut a conic in pairs of points t_1, t_2 which belong to a quadratic involution.[1]

Many properties of the involution are now geometrically apparent. Thus we see that the involution has two double points for the double points are plainly cut out by the tangent lines of the pencil, *i. e.*, they lie on the polar line of x. We have thus through the polar system *a (1, 1) correspondence between points of the plane and quadratics, viz.*, to the point x corresponds the quadratic giving the double points of the involution determined by x.

We can use this correspondence to find the polar line of a point a. The quadratic corresponding to a is from (4)

$$a_3t^2 - a_2t + a_1 = 0.$$

[1] This theorem can be deduced from the correspondence between lines and quadratics. For to a pencil of lines corresponds a pencil, *i. e.*, an involution of quadratics.

It can also be derived geometrically. For any point t_1 of the conic determines with a fixed point x of the plane a line which cuts the conic in a second point t_2. But the line xt_2 cuts again in the point t_1, hence the relation between t_1 and t_2 is involutory.

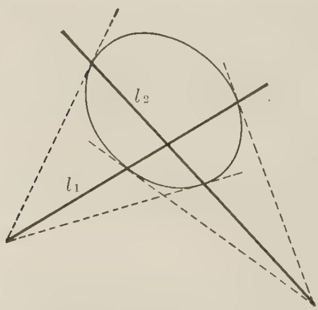
And by (2) and (3) the line corresponding to this quadratic, *i. e.*, the polar line of a is

$$2a_3x_1 - a_2x_2 + 2a_1x_3 = 0.$$

Again the involution is determined by two pairs of conjugate points, the two double points, or one double point and one conjugate pair. For in either case we have two lines of the pencil which of course determine the pencil and the associated involution.

Further the involution will be elliptic, parabolic, or hyperbolic according as the point x is inside, on or outside the conic. When x is on the conic it is an element in every pair, the involution is singular and the double points coincide at x. We are thus led geometrically to the theorem: The common point of two quadratics is a repeated point of their Jacobian. Or the resultant of two quadratics is the discriminant of their Jacobian. (§102, 2° and Cor.)

131. The geometric interpretation of a system of quadratics on the conic consists chiefly in tracing the correspondence between the quadratics and their associated system of lines. We shall use for the quadratics the notation of §113 (except that the variable is understood to be t instead of x) while the line corresponding to the quadratic f_i will be denoted by l_i, that corresponding to J_{12} by l_{12}, etc.



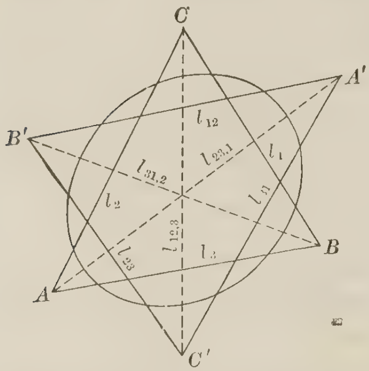
If two quadratics f_1, f_2 are apolar each represents a pair of points in the involution of which the other represents the double points. Hence the line l_1 passes through the pole of the line l_2 and *vice versa*. In other words

Apolar quadratics are cut out by a pair of conjugate polar lines (§74).

We shall now indicate in parallel columns the interpretation of the system of quadratics in terms of their corresponding lines, an interpretation which the student will readily verify. This table exhibits an interesting connection between the binary system of quadratics and the ternary system of the conic and lines in the plane. A noteworthy feature revealed by the correspondence is that *invariant relations in the binary system imply projective relations in the ternary as well as in the binary system.*

Binary System	*Ternary System*
(*a*) Quadratic f.	Line l.
(*b*) If $D_{11} = 0$ f_1 is a square.	l_1 is tangent to the conic.
(*c*) If $D_{12} = 0$, f_1 and f_2 are apolar.	l_1 and l_2 are conjugate polar lines.
(*d*) If $R_{12} = 0$, f_1 and f_2 have a common factor.	l_1 and l_2 meet on the conic.
(*e*) If $I_{123} = 0$, f_1, f_2, f_3 are in an involution.	l_1, l_2, l_3 meet in a point.
(*f*) J_{12} represents the double points of the involution determined by f_1, f_2.	l_{12} is the polar line of the point of intersection of l_1, l_2.
(*g*) J_{23}, J_{31}, J_{12} are the double points of the involutions determined by the quadratics in pairs.	l_{23}, l_{31}, l_{12}, form the polar triangle of l_1, l_2, l_3.
(*h*) If $D_{23} = 0$, $D_{31} = 0$, $D_{12} = 0$, f_1, f_2, f_3, are mutually apolar, *i. e.*, each represents the double points of the involution determined by the other two.	l_1, l_2, l_3 form a self-polar triangle.

As an application of this correspondence we shall prove that a triangle and its polar triangle are perspective (§74, Ex. 12, or §127, Ex. 14). According to the notation of the present section the sides of the two triangles may be designated (by (g)) l_1, l_2, l_3 and l_{23}, l_{31}, l_{12}. Denoting the opposite vertices of the two triangles by A, B, C and A', B', C' respectively, we are to prove the triangles $\begin{pmatrix} A & B & C \\ A' & B' & C' \end{pmatrix}$ perspective.



Since $J_{23,1}$ is apolar to J_{23} and f_1, the line $l_{23,1}$ will pass through the poles (A, A') of l_{23} and l_1, *i. e.*, $l_{23,1} \equiv AA'$. Likewise BB' will be $l_{31,2}$ and CC' will be $l_{12,3}$.

Now (§115, Ex. 16 (b))

$$\begin{aligned} J_{12,3} &= D_{31}f_2 - D_{23}f_1 \\ J_{31,2} &= D_{23}f_1 - D_{12}f_3 \\ J_{23,1} &= D_{12}f_3 - D_{31}f_2 \end{aligned}$$

whence, adding

$$J_{12,3} + J_{31,2} + J_{23,1} \equiv 0.$$

Thus the three quadratics, being linearly dependent, belong to an involution and consequently their corresponding lines AA', BB' and CC' belong to a pencil. Q. E. D.

The geometric results of this section have been obtained as a translation of the theory of quadratics. We might with equal effectiveness reverse the procedure as in §130 and use our knowledge of the geometry of the conic to derive invariant properties of quadratics. For example

since the relation between polar triangles is mutual (§74, Ex. 11), it follows from (g) that l_1, l_2, l_3 is the polar triangle of l_{23}, l_{31}, l_{12}. Translated into algebra this says that if we form the Jacobians of J_{23}, J_{31}, J_{12} in pairs we shall recover the original quadratics f_i as can be verified by Ex. 16(c) §115.[1]

Neither process (translating the algebra into geometry or translating the geometry into algebra) should be neglected but each should be used as supplementary to the other. For here as elsewhere a property overlooked in the one theory will frequently be suggested by the other. Thus from the first equation above we learn that since $J_{12,3}$ is expressible as a linear combination of f_1 and f_2, $l_{12,3}$ belongs to the pencil determined by l_1 and l_2. But it is geometrically obvious (see figure) that it is also a member of the pencil determined by l_{23} and l_{31}. Hence we deduce the algebraic theorem unnoticed before that $J_{12,3}$ can be expressed as a linear function of J_{23} and J_{31}.

132. The system of quadratics in the parametric equations.—The intimate contact between the algebra of quadratics and the geometry of the conic would be anticipated from the very existence of the parametric equations. And in virtue of this parametric representation the conic becomes the peculiar instrument for the geometric interpretation of a system of three quadratics. Thus when the three quadratics are taken as the ternary coördinates of a point of the conic, their three Jacobians in pairs represent the ternary coördinates of the tangent at the point.

If the quadratics are mutually apolar they satisfy an identical quadratic relation (§115, Ex. 8)

$$k_1f_1^2 + k_2f_2^2 + k_3f_3^2 \equiv 0, \tag{1}$$

[1] This also follows from the fact that the parametric point equations can be recovered from the parametric line equations by the Jacobian process.

whence substituting the x's we obtain the equation of the conic in the form

$$k_1x_1^2 + k_2x_2^2 + k_3x_3^2 = 0. \tag{2}$$

Therefore *the equation of a conic referred to a self-polar triangle reduces to a sum of three squares.*

In §123 we stipulated that the three quadratics should be linearly independent. What happens if this condition is violated? If the quadratics are not linearly independent, *i. e.*, if $I_{123} = 0$ so that the quadratics belong to an involution, the x's cannot be linearly independent and the point x is therefore confined to a line. The conic is in fact a line repeated.

In particular if the quadratics contain a common factor they are not linearly independent[1] (§89, Ex. 12). If the common factor is $t - t_1$, the equations of the conic take the form

$$x_i = (t - t_1)(\alpha_i t + \beta_i), \qquad i = 1, 2, 3. \tag{3}$$

If $t = t_1$, the point x is arbitrary. Otherwise, dividing out the common factor since only the ratios of the x's are significant, the conic degenerates to a line whose parametric equations are

$$x_i = \alpha_i t + \beta_i. \tag{4}$$

Or the conic may be regarded as line (4) and an arbitrary line of the pencil on the point (of (4)) whose parameter is t_1.

EXERCISES

1. Derive the parametric line equations of the conic (1) §123.

2. Write the parametric line equations of the following curves:

(*a*) $x_1 = t^3, x_2 = t^2, x_3 = 1,$
(*b*) $x_1 = 3t^2, x_2 = 3t, x_3 = t^3 + 1,$
(*c*) $x_1 = t^3 + 1, x_2 = t^4 + t, x_3 = t^2,$

[1]For the common factor may be taken as $t = 0$, when $c_1 = c_2 = c_3 = 0$ and I_{123} unmistakeably vanishes.

(*d*) $x_1 = t^5 + 5t^2,\ x_2 = 5t^3 + 1,\ x_3 = t^4 + t,$
(*e*) $x_1 = t^3 + 3t,\ x_2 = 3t^5 - 1,\ x_3 = -5t^3.$

What is the class of each curve?

3. Find by inspection or by Richmond's method the ternary equations of the curves of Ex. 2. *Ans.* (*c*) $(x_1^3 + x_2^3)x_3 - x_1^2x_2^2x_3 = 0.$

4. Show that the conic $x_1 = 3,\ x_2 = 3t^2,\ x_3 = 2t$, and the cubic (*b*) Ex. 2 have three contacts. (Substitute the values of the x's from the cubic in the ternary point equation of the conic.) Find the parameters (on the cubic) of the common lines of the two curves.

5. Show that the curve of Ex. 3 and $x_1 = 2t^3 + 1,\ x_2 = t^4 + 2t,\ x_3 = t^2$ have eight contacts.

6. Show that a rational curve whose point coördinates are defined by binary n-ics is of order n and that the class of the curve is in general $2n - 2$.

7. A curve which is rational in points is also rational in lines. (The partial derivatives in (9) §128 are all rational functions.)

8. The general rational (plane) curve of order n contains $3n - 1$ essential constants, *viz.*, n for each binary form in the parametric equations diminished by one because of the relation implied by the ternary equation. Thus the general rational curve falls short of the general plane curve by $\frac{1}{2}(n - 1)(n - 2)$ constants. In other words it is $\frac{1}{2}(n - 1)(n - 2)$ conditions on the general plane curve to be rational. The conditions are in fact, that is possess precisely this number (the maximum number) of double points or their equivalent in multiple points.

9. The ternary equation of any rational curve can be found by dialytic elimination as a $3n$-row determinant. (Multiply the equations in turn by $t, t^2 \ . \ . \ . \ t^{n\ 1}$.)

10. Identify the formula ((4) §128) for finding the tangent line with the usual calculus form of the equation: $y - y' = dy'/dx'$ $(x - x')$. (Write x for x_1/x_3 and y for x_2/x_3.)

11. Every coefficient in the ternary point and line equations ((6) §123 and (1) §127) is an invariant of the three quadratics.

12. Interpret geometrically as in §131 the following invariant relations in the system of quadratics:

$$(a)\ D_{12,3} \equiv I_{123} = 0$$
$$(b)\ D_{12,34} \equiv I_{12,3,4} = 0$$
$$(c)\ J_{12,34} \equiv f_2I_{134} - f_1I_{234} \equiv f_3I_{412} - f_4I_{123}.$$

13. Deduce the fundamental theorem of poles and polars (I', §74) from the geometry of apolar quadratics on the conic.

14. In §131 we remarked that $J_{12,3}$ could be expressed as a linear function of J_{23} and J_{31}. Find this expression.

$$Ans. \quad \sqrt{-D_{33}/2}\, J_{12,3} \equiv D_{31}J_{23} + D_{23}J_{31}.$$

15. From the map equation of the conic in lines $x_1J_{23} + x_2J_{31} + x_3J_{12} = 0$, obtain the identity $f_1J_{23} + f_2J_{31} + f_3J_{12} \equiv 0$. Or from the identity obtain the map equation.

16. Find the quadratic corresponding to the axis of perspection of the triangles in §131. (This quadratic gives the double points of the involution.) *Ans.* $\dfrac{J_{23}}{D_{23}} + \dfrac{J_{31}}{D_{31}} + \dfrac{J_{12}}{D_{12}} = 0.$

17. Show that the quadratic of Ex. 16 can be written in the form

$$\frac{f_1}{\delta_{23}} + \frac{f_2}{\delta_{31}} + \frac{f_3}{\delta_{12}} = 0,$$

where δ_{ij} is the cofactor of D_{ij} in the determinant of D's in Ex. 14, §115.

18. If the conic is in the canonical form (§124) find the involution cut out by lines on the vertex u_2. (Denote corresponding points in the involution by t and t'.) Find the center of the pencil of lines which cut out the involution (*a*) containing the pairs t_1, t'_1 and t_2, t'_2, (*b*) of which $a_it^2 + 2b_it + c_i$, $i = 1, 2$, represent conjugate points, (*c*) whose double points are given by $at^2 + 2bt + c = 0$, (*d*) with double points ± 1.

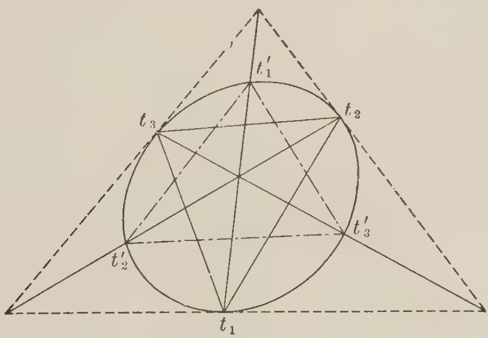
19. Prove Pascal's theorem, using the parametric representation of the conic.

20. Verify the statement (§132) that when the three quadratics in the equation of a conic belong to an involution the point equation ((1), §127) is the square of a line. (If the relation among the quadratics is $x_3 \equiv k_1x_1 + k_2x_2$, the conic is $(k_1x_1 + k_2x_2 - x_3)^2 = 0$.)

21. If the binary forms of order n in the parametric equations of a rational curve have a common factor $(t - t_1)^m$, the curve reduces to a rational curve of order $n - m$.

22. The quartic curve in the example of §128 is of class 6. This class sextic should be of order 10 (Ex. 6, dual). It must, however represent the original quartic. How do you explain the paradox? (Calculate the parametric point equations from the line equations and apply Ex. 21.) Discuss the general case for the rational curve of order n. What conclusion can you draw concerning the three Jacobians in pairs of the Jacobians in pairs of three binary forms of order n?

133. The binary cubic on the conic.—The conic also affords an excellent geometric interpretation for the covariants of a binary cubic. The cubic itself will represent three points say t_1, t_2, t_3 and the cubicovariant three corresponding points t_1', t_2', t_3' such that t_i, t_i' are pairs in a quadratic involution whose double points are given by the Hessian (§115). These three pairs of conjugate elements will lie on lines of a pencil the polar of whose center cuts out the points H. In other words



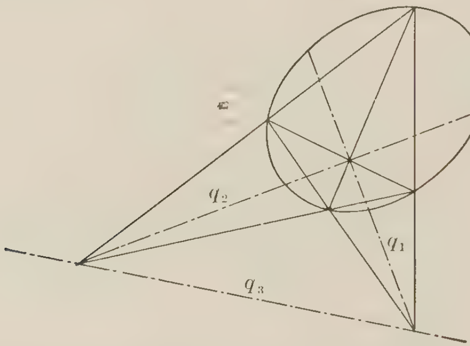
The cubic and the cubic covariant points are vertices of the perspective triangles $\begin{pmatrix} t_1 & t_2 & t_3 \\ t_1' & t_2' & t_3' \end{pmatrix}$ *whose axis cuts the conic in the Hessian points.*

Given the cubic points t, to construct the cubicovariant points t' we recall that t_1, t_1' are harmonic conjugates of t_2, t_3. Hence the line joining t_1 to the pole of $\overline{t_2t_3}$ cuts the conic again in the point t_1'. But the poles of the sides $\overline{t_it_j}$ are simply the vertices of the circumscribed triangle touching at the points t. The theory of the binary cubic thus furnishes incidentally another proof that three points on a conic and the tangents at the points are triangles in perspective position.

The syzygetic pencil of a cubic and its cubicovariant will represent a singly infinite system of triangles inscribed in the conic. Since every cubic in the pencil has the same Hessian, *every triangle of the system is perspective from the same axis with the triangle touching at its vertices.*

134. The geometry of the binary quartic on a conic is a geometry of quadrangles. First of all

The quartic represents the vertices of an inscribed quadrangle whose diagonal triangle cuts out the six points given by the sextic covariant.



For to the three pairs of opposite sides of the quadrangle correspond three pairs of quadratics which determine the three quadratic involutions associated with the binary quartic. The double points of these involutions which are the quadratic factors of the sextic covariant are cut out by the polars of the diagonal points (intersections of pairs of opposite sides). But the diagonal triangle is self-polar so that its sides correspond to the quadratic factors of J. Q. E. D.

Again the Hessian represents four points with the same diagonal triangle as the quadrangle f. Indeed every member of the syzygetic pencil $f + \lambda H$ represents a quad-

rangle with the same diagonal triangle as f. For every member of the syzygetic pencil has the same sextic covariant (§120). Or we may say

Any self-polar triangle represented by the sextic J is diagonal triangle to a single infinity of inscribed quadrangles, namely the quadrangles represented by the syzygetic pencil $f + \lambda H$, where f is any quartic apolar to J.

Correspondence between conics and binary quartics on the fundamental conic. Take the fundamental (fixed) conic as

$$K\colon\ x_1 = t^2, \quad x_2 = 2t, \quad x_3 = 1, \tag{1}$$

and the general conic in the form

$$ax_1^2 + bx_2^2 + cx_3^2 + 2fx_2x_3 + 2gx_3x_1 + 2hx_1x_2 = 0. \tag{2}$$

Then substituting the values of the x's from (1) in (2) we have as the equation giving the parameters of the points of intersection of the two conics

$$at^4 + 4ht^3 + (2g + 4b)t^2 + 4ft + c = 0. \tag{3}$$

Now setting $b = g = k$ in (2) and (3) we obtain the *conic and corresponding binary quartic*

$$\begin{aligned} U\colon ax_1^2 + 2hx_1x_2 + k(2x_3x_1 + x_2^2) \\ + 2fx_2x_3 + cx_3^2 = 0 \end{aligned} \tag{4}$$

$$f\colon\ at^4 + 4ht^3 + 6kt^2 + 4ft + c, \tag{5}$$

i. e., the binary quartic f corresponds to the conic U which cuts K in the four points represented by f.

While f is a general binary quartic, the corresponding conic U is subject to one condition. If in the ternary line form $u_2^2 - u_3u_1$ of K we replace u_i by $\partial/\partial x_i$ and operate on (2) the result will be found to vanish if $b = g$ which was just the condition that (2) reduce to U. Two conics related as U and K are said to be *apolar*. (See below, §141.)

The correspondence just established connects the theory of the binary quartic with that of a pair of apolar conics. Any invariant specialization of f implies at once a projective property of f and a projective relation between the two conics. Thus if the discriminant of f vanishes, two points of f coincide and U and K have contact. If $I_3 = 0$, f factors into a pair of apolar quadratics and U breaks up into a pair of lines conjugate with respect to K.

The Hessian of the quartic represents on K the four contacts of common lines of U and K.

To prove this we resort to the canonical form of f

$$t^4 + 6kt^2 + 1 \tag{6}$$

which corresponds to the conic

$$x_1^2 + kx_2^2 + x_3^2 + 2kx_3x_1 = 0. \tag{7}$$

The line equation of (7) is (§15, Ex. 5)

$$ku_1^2 + (1 - k^2)u_2^2 + ku_3^2 - 2k^2u_3u_1 = 0. \tag{8}$$

Combining (8) with the line equations of K

$$u_1 = 1, \quad u_2 = -t, \quad u_3 = t^2, \tag{9}$$

we find for the quartic giving the (contacts of the) common lines of (8) and (9)

$$kt^4 + (1 - 3k^2)t^2 + k, \tag{10}$$

which is the canonical form of the Hessian. Q. E. D.

There is of course a conic U' corresponding to the Hessian.[1] The pencil of conics $U + \lambda U' = 0$ cut K in the system of quadrangles represented by $f + \lambda H$.

[1] This is the conic F of Salmon, *Conic Sections*, §§334, 378. It is the locus of points from which the tangents to the two conics form a harmonic pencil, and passes through the eight contacts of common lines of U and K.

For an excellent account of the complete system of invariant forms associated with two conics consult Salmon, Chap. XVIII.

EXERCISES

These exercises refer to the geometry on a conic where the point is to be interpreted as an element in the binary domain and the conic is in canonical form.

1. Show geometrically that if a binary cubic have a double point the Hessian has the same double point and the cubicovariant has it for a triple point.

2. If t_1, t_2, t_3 are three points on a conic find the ternary coördinates of the pole of the line corresponding to the Hessian. (The cubic giving the three points is $t^3 - s_1t^2 + s_2t - s_3$.)

3. Verify the following construction for the polar of a point t on a conic with respect to three points a_1, b_1, c_1 on the conic: Circumscribe a triangle to touch at a_1, b_1, c_1 and denote the vertices opposite these points by A_1, B_1, C_1. Draw tA_1, tB_1, tC_1, cutting the conic respectively in points a_2, b_2, c_2. Then the three pairs of points like a_1, a_2 and b_1, c_1 determine three quadratic involutions. Prove that the double points of these three involutions are themselves pairs in an involution whose double points are the polar pair of t with respect to a_1, b_1, c_1. (It will suffice to prove the construction for the points $0, 1, \infty$.)

4. Find the three quadratic involutions determined by the four points $t, -t, 1/t, -1/t$ and describe the effect of each on the set of points.

5. Given the points $0, 1, \infty$ on a conic and one point t of a binary quartic (in canonical form) construct geometrically the remaining points of the quartic.

6. Show geometrically that if a binary quartic have a double point the sextic covariant has the point as a quintuple point. What becomes of the three involutions and their double points?

7. Show that there can be no more than ∞^1 inscribed quadrangles with the same self-polar triangle. (There are only two linearly independent quartics apolar to a sextic.)

8. When the binary quartic on a conic is taken in canonical form, find the conic corresponding to (*a*) the Hessian, (*b*) the Steinerian, (*c*) the Hessian of the Hessian of f, (*d*) the quartic which vanishes when $f \equiv H$.

9. What is the condition that the two conics U and K (§134) should *osculate, i. e.*, meet in three consecutive points? that they meet in four consecutive points?

10. Show that the I_2 of a binary quartic f may be found by substituting differential symbols in the line equation of the conic cor-

responding to f and operating on the point equation of K. (This is the invariant Θ' of the two conics. See Salmon, *l. c.*)

11. Show geometrically that a double point of a binary quartic is a double point of the Hessian.

12. Discuss the geometry of the system of quadratics, the binary cubic and the binary quartic on a conic (§§131, 133, 134) when the fundamental forms represent lines and not points of the conic.

13. Show that the cubic curve $ax_1^3 + bx_2^3 + cx_3^3 + 3dx_1^2x_2 + 3ex_1x_2^2 + 3fx_1^2x_3 + 3gx_1x_3^2 + 3hx_2^2x_3 + 3ix_2x_3^2 + 6jx_1x_2x_3 = 0$ will correspond to the binary sextic $at^6 + 6dt^5 + 15ft^4 + 20bt^3 + 15gt^2 + 6it + c$, if $f = e$, $b = j$, $g = h$ which are the conditions that K be apolar to the cubic (see below, §141). Thence find the correspondence between the cubic curve and the square of the binary cubic $\alpha t^3 + 3\beta t^2 + 3\gamma t + \delta$. Show that the cubic curve which corresponds to the binary cubic has contacts with K at points given by the binary cubic.

Part II. The Conic as a Ternary Form

135. We shall write the ternary quadratic, *i. e.*, the general conic in the standard form[1]

$$f(x_1, x_2, x_3) \equiv a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{23}x_2x_3 + 2a_{31}x_3x_1 + 2a_{12}x_1x_2 = 0 \quad (1)$$

where the x's are projective coördinates and where $a_{ik} = a_{ki}$.

The first fundamental problem is *to determine the points in which an arbitrary line meets the conic.* If (y_1, y_2, y_3) and (z_1, z_2, z_3) are any two points on the line the parametric equations of the line are (§55)

$$\begin{aligned} x_1 &= z_1t + y_1 \\ x_2 &= z_2t + y_2 \\ x_3 &= z_3t + y_3. \end{aligned} \quad (2)$$

[1] The subscript notation indicates at a glance the term to which each coefficient belongs. An alternative form of the equation easier to write is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

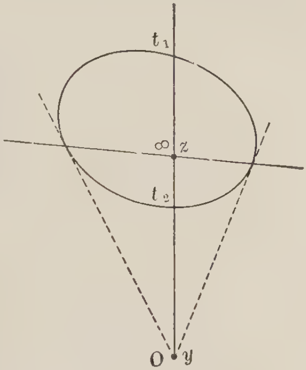
Replacing the x's in (1) by their values from (2) we have by Taylor's theorem or direct substitution
$f(z_1t + y_1, z_2t + y_2, z_3t + y_3) \equiv$

$$f(z)\, t^2 + \left(y \frac{\partial}{\partial z}\right) f(z)t + f(y) = 0, \quad (3)$$

where

$$\left(y \frac{\partial}{\partial z}\right) \equiv y_1\frac{\partial}{\partial z_1} + y_2\frac{\partial}{\partial z_2} + y_3\frac{\partial}{\partial z_3}. \quad (4)$$

Considered as an equation in t, (3) is a quadratic which gives the parameters (on the line) of the two points cut out by the conic.



Polar Properties. Now the double ratio of four points on a line is equal to the double ratio of their parameters (§55). Hence denoting the roots of (3) by t_1, t_2 the double ratio of the points y, z and the points $y + t_1z$, $y + t_2z$ in which their junction meets the conic is $(0\ \infty \mid t_1t_2) = t_1/t_2$. In particular if $t_1 + t_2 = 0$ the four points are harmonic. The condition for this is from (3)

$$\left(y \frac{\partial}{\partial z}\right) \equiv \left(y_1\frac{\partial}{\partial z_1} + y_2\frac{\partial}{\partial z_2} + y_3\frac{\partial}{\partial z_3}\right)f(z) = 0. \quad (5)$$

If y is held fast while z is allowed to vary then (5) is the equation of a line,—the locus of points harmonically separated from y by the conic.[1] In other words (5) *is the equation of the polar line of y with respect to the conic.*

Equally for constant z and variable y, the harmonic relation being mutual, (5) is the polar of z as to the conic.

[1] That is, by the pairs of points in which zy cuts the conic.

It follows or it can be seen in various ways that the equation of the polar line is symmetrical in y and z, *i. e.*,

$$\left(y\frac{\partial}{\partial z}\right)f(z) \equiv \left(z\frac{\partial}{\partial y}\right)f(y).$$

We have thus a proof of the fundamental theorem:

1°. *If y lies on the polar of z, then z lies on the polar of y.*

The condition that a point y lie on its own polar $\left(z\frac{\partial}{\partial y}\right)f(y) = 0$ is $\left(y\frac{\partial}{\partial y}\right)f(y) = 0$. But by Euler's theorem for homogeneous functions (§97)

$$\left(y\frac{\partial}{\partial y}\right)f(y) \equiv 2f(y).$$

Hence

2°. *The locus of a point which lies on its own polar with respect to a conic is the conic itself.*

If y is on the conic $f(y) = 0$ and one root t_1 of the quadratic (3) is zero. If in addition $\left(z\frac{\partial}{\partial y}\right) f(y) = 0$, $t_2 = 0$ and the line $\overline{yz}$ is tangent to the conic. Therefore if y is on the conic its polar line is tangent to the curve and in virtue of 2° the contact is at y. Or

3°. *The polar of a point on the conic is the tangent at the point.*

If the discriminant of the t-equation vanishes, *i. e.*, if

$$4f(z)\,f(y) - \left\{\left(z\frac{\partial}{\partial y}\right)f(y)\right\}^2 = 0 \tag{6}$$

the line $\overline{zy}$ is tangent to the curve. It follows that if y is fixed z can move only along a tangent from y to the conic. Hence (6) *which represents a locus (of z) of the second order is the ternary equation of the pair of tangents from y.*

136. Line equation of the conic.—At full length the polar of y with respect to the conic $f(x_1, x_2, x_3) = 0$ may be written, with x instead of z as the variable coördinate,

$$\begin{aligned}\tfrac{1}{2}\left(x\frac{\partial}{\partial y}\right)f(y) \equiv (a_{11}y_1 + a_{12}y_2 + a_{13}y_3)x_1 \\ + (a_{21}y_1 + a_{22}y_2 + a_{23}y_3)x_2 \\ + (a_{31}y_1 + a_{32}y_2 + a_{33}y_3)x_3 = 0.\end{aligned} \qquad (1)$$

Whence the coördinates of the polar line of y are

$$\begin{aligned}\sigma u_1 &= a_{11}y_1 + a_{12}y_2 + a_{13}y_3 \\ \sigma u_2 &= a_{21}y_1 + a_{22}y_2 + a_{23}y_3 \\ \sigma u_3 &= a_{31}y_1 + a_{32}y_2 + a_{33}y_3.\end{aligned} \qquad (2)$$

If the determinant

$$A \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$$

we may solve equations (2) for y thus

$$\begin{aligned}\rho y_1 &= A_{11}u_1 + A_{21}u_2 + A_{31}u_3 \\ \rho y_2 &= A_{12}u_1 + A_{22}u_2 + A_{32}u_3 \qquad A_{ik} = A_{ki} \\ \rho y_3 &= A_{13}u_1 + A_{23}u_2 + A_{33}u_3\end{aligned} \qquad (3)$$

where A_{ik} is the cofactor of a_{ik} in A. Equations (3) give the coördinates y of the pole of the line u.

Now the locus of a line which passes through its own pole is the conic itself in lines (§135, 2°, dual). Hence to find the line equation we merely ask that a line and its polar point be incident. Let the equation of the line u be

$$(ux) \equiv u_1x_1 + u_2x_2 + u_3x_3 = 0. \qquad (4)$$

Then substituting from (3) the coördinates of its polar point y we have for the condition that the line pass through its pole, *i. e.*, the line equation of the curve

$$\begin{aligned}(uy) \equiv A_{11}u_1^2 + A_{22}u_2^2 + A_{33}u_3^2 + 2A_{23}u_2u_3 + 2A_{31}u_3u_1 \\ + 2A_{12}u_1u_2 = 0.\end{aligned} \qquad (5)$$

Or again the coördinates of the polar line of y are the expressions in (2) and the condition that y lie on its own polar (4) is

$$u_1y_1 + u_2y_2 + u_3y_3 = 0. \tag{6}$$

Thence eliminating y_1, y_2, y_3 and σ from equations (2) and (6) we obtain the line equation in the form of the symmetrical *bordered* determinant:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & u_1 \\ a_{21} & a_{22} & a_{23} & u_2 \\ a_{31} & a_{32} & a_{33} & u_3 \\ u_1 & u_2 & u_3 & 0 \end{vmatrix} = 0. \tag{7}$$

EXERCISES

In these exercises the notation is the same as that of §§135–6 unless otherwise stated.

1. Write in full the equation of the polar line of (y_1, y_2, y_3) with respect to the conic f. Thus to find the polar line of y (or the tangent at y) replace x_i^2 by x_iy_i and $2x_ix_k$ by $x_iy_k + x_ky_i$. The equation of f is then said to be polarized once with respect to y.

2. Find the equation of the polar line of (or tangent at) (x_1, y_1) for the following conics. (Write the equations in homogeneous Cartesian coördinates.)

(*a*) $x^2/a^2 \pm y^2/b^2 = 1$
(*b*) $y^2 = 4ax$
(*c*) $xy = k$
(*d*) $(1 - e^2)x^2 + y^2 - 2px + p^2 = 0.$

3. If $y = x$ in the equation of the polar line of y, the equation reduces to f itself. What is the significance?

4. Find the polars of the vertices of the triangle of reference with respect to the conic f. Show that this polar triangle is perspective with the reference triangle. Find the center and axis of perspection and show that they are pole and polar. Dualize.

5. Find the equations of the three pairs of tangents to f from the vertices of the reference triangle by each of the two methods: (*a*) Apply formula (6) §135; (*b*) write the line equation of f, set $u_i = 0$ and change from line to point coördinates by the substitution (§47)

$u_i/u_k = x_k/-x_j$. *Ans.* Three equations like $A_{33}x_2^2 - 2A_{23}x_2x_3 + A_{22}x_3^2 = 0$.

6. From Ex. 5, the condition that the three pairs of lines $a_1x_2^2 + b_1x_2x_3 + c_1x_3^2 = 0$, $a_2x_3^2 + b_2x_3x_1 + c_2x_1^2 = 0$, $a_3x_1^2 + b_3x_1x_2 + c_3x_2^2 = 0$ touch a conic is $a_1a_2a_3 = c_1c_2c_3$.

7. The polars of a point with respect to the conics of a pencil $f_1 + kf_2 = 0$ are lines of a pencil.

8. If a point move on a line its polar lines with respect to two conics intersect in pairs in the points of a conic (Ex. 9, §74). Find the equation of this conic. (Let the conics be $f_1 \equiv a_{11}x_1^2 + \ldots$ and $f_2 \equiv b_{11}x_1^2 + \ldots$ and let (y_1, y_2, y_3) and (z_1, z_2, z_3) be any two points. Then $y_i + \lambda z_i$ will be coördinates of a variable point on the line yz. The polars of this point with respect to each conic will be a pencil. Make the pencils projective and eliminate λ (§63).)

9. Show that the polar of the center of a conic is the line at infinity.

10. Find the coördinates of the center (pole of $\mathcal{L}$) of the conic

$$ax^2 + 2hxy + by^2 + 2fy + 2gx + c = 0.$$

11. Find the asymptotes (as a line pair) of the conic in Ex. 10. (The asymptotes are tangents from the center.)

12. Find the coördinates of the center and the equations of the asymptotes of the conics

$$3x^2 - 6xy + 6x + 12y - 26 = 0$$
$$3x^2 + 7xy + 4y^2 - 2x - 3y - 6 = 0$$
$$60x^2 + 132xy + 72y^2 + 2x + 3y - 10 = 0.$$

13. Find the equation of the pair of tangents to $y^2 = 4ax$ from a point on the directrix and show that they are perpendicular.

14. Expand the determinant (7) (§136) and identify this form of the line equation with (5). State a rule for forming the coefficient of u_iu_k by striking out rows and columns of the determinant (i may be equal to k.)

15. Write the line equations of the conics in Exs. 2 and 10.

16. Dualize §136 and obtain formulas for deriving the point equation of a conic from the line equation.

17. Write the point equation of (5) §136 and show that it is the original conic. (Make use of Ex. 1, §62.)

18. Write in full the equation and the coördinates of the pole of the line (v_1, v_2, v_3) with respect to the line conic

$$\varphi \equiv a_{11}u_1^2 + a_{22}u_2^2 + a_{33}u_3^2 + 2a_{23}u_2u_3 + 2a_{31}u_3u_1 + 2a_{12}u_1u_2 = 0.$$

19. The three pairs of lines from the vertices of the triangle of reference to the points in which the opposite sides cut a conic touch a second conic. Find the equation of this conic.

Solution. Setting $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ in turn we obtain as the equations of the three pairs of lines in question (§29)

$$\begin{aligned} a_{22}x_2^2 + 2a_{23}x_2x_3 + a_{33}x_3^2 &= 0 \\ a_{33}x_3^2 + 2a_{31}x_3x_1 + a_{11}x_1^2 &= 0 \\ a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 &= 0 \end{aligned} \tag{1}$$

which touch a conic in virtue of Ex. 6. The equation of the conic is best obtained as a line conic. Thus transforming (1) to line coördinates (§47) we have, setting $x_i/x_k = u_k/-u_i$

$$\begin{aligned} a_{22}u_3^2 - 2a_{23}u_2u_3 + a_{33}u_2^2 &= 0 \\ a_{33}u_1^2 - 2a_{31}u_3u_1 + a_{11}u_3^2 &= 0 \\ a_{11}u_2^2 - 2a_{12}u_1u_2 + a_{22}u_1^2 &= 0. \end{aligned} \tag{2}$$

Now multiplying equations (2) respectively by a_{11}, a_{22}, a_{33} it is clear from the symmetry that the required conic is $a_{22}a_{33}u_1^2 + a_{33}a_{11}u_2^2 + a_{11}a_{22}u_3^2 - 2a_{11}a_{23}u_2u_3 - 2a_{22}a_{31}u_3u_1 - 2a_{33}a_{12}u_1u_2 = 0$. For on setting u_1, u_2, u_3 respectively equal to zero equations (2) are recovered. Dualize the statement and proof of this theorem.

20. What do equations (2), Ex. 19, in themselves represent?

21. Show that the following sets of line pairs touch conics and find their equations

$$(a)\quad \begin{aligned} a_{33}x_2^2 + 2a_{23}x_2x_3 + a_{22}x_3^2 &= 0 \\ a_{11}x_3^2 + 2a_{31}x_3x_1 + a_{33}x_1^2 &= 0 \\ a_{22}x_1^2 + 2a_{12}x_1x_2 + a_{11}x_2^2 &= 0 \end{aligned} \qquad (b)\quad \begin{aligned} x_2^2 + 2ax_2x_3 + \omega x_3^2 &= 0 \\ \omega x_3^2 + 2bx_3x_1 + \omega^2x_1^2 &= 0 \\ \omega^2x_1^2 + 2cx_1x_2 + x_2^2 &= 0. \end{aligned}$$

22. Dualize §135 and interpret geometrically the dual equations considered.

23. Find the condition that the pair of lines

$$a_1x_1 + b_1x_2 + c_1x_3 = 0 \text{ and } a_2x_1 + b_2x_2 + c_2x_3 = 0$$

be conjugate with respect to the conic f. Find the condition for the lines $x_1x_2 = 0$.

24. Find the conditions that the three lines

$$a_ix_1 + b_ix_2 + c_ix_3 = 0, \qquad i = 1, 2, 3$$

form a triangle self-polar to f.

25. By adding another variable extend the theory of §135–6 to the quaternary form of the second order $F(x_1, x_2, x_3, x_4) \equiv \Sigma a_{ik}x_ix_k \equiv$

$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{44}x_4^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{14}x_1x_4 + 2a_{23}x_2x_3 + 2a_{24}x_2x_4 + 2a_{34}x_3x_4$, $a_{ik} = a_{ki}$. $F = 0$ then represents a quadric surface or *quadric*. The discussion should include all formulas, definitions and theorems. In particular show that

1°. An arbitrary line cuts the quadric in two points.

2°. The locus of points harmonically separated from a point y by the quadric is a plane (locus of first order), the *polar plane* of y.

3°. If y lies on the polar plane of z, then z lies on the polar plane of y.

4°. The locus of a point which lies on its own polar plane with respect to a quadric is the quadric itself.

5°. The polar plane of a point on the quadric is the tangent plane at the point.

6°. The discriminant of the (new) t-equation represents for variable z a quadric surface which is the locus of the tangent lines from y to F, *i. e.*, the *tangent cone* with vertex y.

Dualize these theorems recalling that in S_3 a point and plane are dual elements and a line is self dual. Find the *plane equation* (condition that a plane touch the quadric) by applying the dual of 4°.

26. Generalize Ex. 25 for S_4 and S_n.

137. Discriminant of the conic.—Suppose now that y is a *double point* of the conic, *i. e.*, that the line $\overline{yz}$ cuts the curve in coincident points y regardless of the position of z.[1] This says, referring to the t-equation, that $f(y) = 0$ and in addition

$$\left(z_1 \frac{\partial}{\partial y_1} + z_2 \frac{\partial}{\partial y_2} + z_3 \frac{\partial}{\partial y_3}\right) f(y) \equiv 0 \tag{1}$$

or

$$\frac{\partial f}{\partial y_1} = 0, \quad \frac{\partial f}{\partial y_2} = 0, \quad \frac{\partial f}{\partial y_3} = 0, \tag{2}$$

[1] The conic must then degenerate into a pair of lines whose intersection is regarded as a double point. The conic is variously described as *degenerate, improper* or *composite*.

Equations (2) are thus necessary conditions for a double point. They are likewise sufficient conditions. For if equations (2) are satisfied (1) is true for every z. In particular it is true for $z = y$ or

$$0 = y_1 \frac{\partial f}{\partial y_1} + y_2 \frac{\partial f}{\partial y_2} + y_3 \frac{\partial f}{\partial y_3} = 2f(y) \tag{3}$$

by Euler's theorem (§97). Hence by (3) and (1) y is on the curve and both roots of the t-equation are zero. In other words the line $\overline{yz}$ for arbitrary z cuts the conic in coincident points at y. Q. E. D.

The conditions for a double point can however be combined into a single condition. For equations (2) when written in full are

$$\begin{aligned} \tfrac{1}{2}\frac{\partial f}{\partial y_1} &\equiv a_{11}y_1 + a_{12}y_2 + a_{13}y_3 = 0 \\ \tfrac{1}{2}\frac{\partial f}{\partial y_2} &\equiv a_{21}y_1 + a_{22}y_2 + a_{23}y_3 = 0 \\ \tfrac{1}{2}\frac{\partial f}{\partial y_3} &\equiv a_{31}y_1 + a_{32}y_2 + a_{33}y_3 = 0. \end{aligned} \tag{4}$$

If these equations are consistent we must have

$$A \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0.$$

Conversely if $A = 0$, equations (4) have a common system of values. Summarizing

A necessary and sufficient condition that a conic have a double point, i. e., that it break up into a pair of lines is $A = 0$.

The determinant A is the *discriminant* of the conic.

138. Conics with vanishing discriminant.—We shall now examine the effect on the foregoing theory when the conic

has a double point. The conditions are if y is the double point

$$\begin{aligned} \tfrac{1}{2}\frac{\partial f}{\partial y_1} &\equiv a_{11}y_1 + a_{12}y_2 + a_{13}y_3 = 0 \\ \tfrac{1}{2}\frac{\partial f}{\partial y_2} &\equiv a_{21}y_1 + a_{22}y_2 + a_{23}y_3 = 0 \\ \tfrac{1}{2}\frac{\partial f}{\partial y_3} &\equiv a_{31}y_1 + a_{32}y_2 + a_{33}y_3 = 0 \end{aligned} \tag{1}$$

or

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0.$$

To find the coördinates of the double point we solve equations (1) in pairs whence

$$y_1 : y_2 : y_3 = A_{11} : A_{12} : A_{13} = A_{21} : A_{22} : A_{23} = A_{31} : A_{32} : A_{33}. \tag{2}$$

Combining these we find that the coördinates of the double point satisfy the following relations

$$A_{11} : A_{22} : A_{33} : A_{23} : A_{31} : A_{12} = y_1^2 : y_2^2 : y_3^2 : y_2y_3 : y_3y_1 : y_1y_2. \tag{3}$$

To find the line equation substitute the values of the A's just found in equation (5) §136 obtaining

$$\begin{aligned} u_1^2y_1^2 + u_2^2y_2^2 + u_3^2y_3^2 + 2u_2u_3y_2y_3 + 2u_3u_1y_3y_1 \\ + 2u_1u_2y_1y_2 = (u_1y_1 + u_2y_2 + u_3y_3)^2 = 0 \end{aligned} \tag{4}$$

which is the square of the equation of the double point. Hence

If a conic break up into a pair of (distinct) lines the line equation represents two coincident points, namely the twice counted intersection of the lines.[1]

Dually if a line conic degenerate into a pair of points the point equation represents the junction of the point pair repeated.

[1] This can be seen geometrically for the conic as an envelope consists of all tangents to the point curve, and the only lines satisfying this condition of (improper) tangency are the lines on the double point.

If however a point conic become a repeated line, say $(a_1x_1 + a_2x_2 + a_3x_3)^2 = 0$, then every A_{ik} is zero and the line equation vanishes identically. Conversely if every A_{ik} is zero the point conic f becomes $(\sqrt{a_{11}}\,x_1 + \sqrt{a_{22}}\,x_2 + \sqrt{a_{33}}\,x_3)^2 = 0$,—obviously a line repeated. Hence

The necessary and sufficient condition that a point conic be a repeated line is that the minor of every element in its discriminant be zero or that the line equation vanish identically.[1]

Polar Properties. Taking the extreme case first, if a conic is a double line $(a_1x_1 + a_2x_2 + a_3x_3)^2 = 0$ the polar of (y_1, y_2, y_3) is $(a_1y_1 + a_2y_2 + a_3y_3)(a_1x_1 + a_2x_2 + a_3x_3) = 0$. Therefore

1°. *If a point conic is a line repeated the polar of any point not on the line is the line itself while the polar of a point on the line is arbitrary.*

If a point conic consist of two distinct lines u and v intersecting at y we have the following theorems:

2°. *The polar of every point in the plane except y passes through the double point.* (*Cf.* §57, (2).)

For the polar of a point z is, x the variable coördinate,

$$z_1\frac{\partial f}{\partial x_1} + z_2\frac{\partial f}{\partial x_2} + z_3\frac{\partial f}{\partial x_3} = 0.$$

Thence substituting y for x

$$\left(z\frac{\partial}{\partial x}\right)f_{x=y} \equiv z_1\frac{\partial f}{\partial y_1} + z_2\frac{\partial f}{\partial y_2} + z_3\frac{\partial f}{\partial y_3} \equiv 0 \text{ (for every } z\text{) by (1),}$$

which proves the theorem.

3°. *The polar of the double point itself is arbitrary.*

For the polar of y

$$x_1\frac{\partial f}{\partial y_1} + x_2\frac{\partial f}{\partial y_2} + x_3\frac{\partial f}{\partial y_3} \equiv 0 \text{ by (1).}$$

[1] This is also geometrically evident since any line in the plane cuts such a conic in two coincident points.

While every point but y has a unique polar, no line would seem to have a determinate pole since the coördinates of the pole (§136, (3)) can no longer be found. But recalling that the conic in lines is y taken twice, we may say

4°. *The pole of any line not on y is y*[1](1°, dual).

5°. *The pole of a line l on y is any point on the harmonic conjugate l' of l as to u and v.*

For every point of l' satisfies the condition for a pole since if x is such a point every line on x cuts l in a point harmonically separated from x by u and v.

139. We are now in a position to give a *projective classification of conics.* We have first the proper conic both in points and lines depending on five essential constants. A point conic may degenerate into (a) two distinct lines when it possesses four constants,—two for each line; (b) a repeated line which contains but two constants. It is thus geometrically evident that the conditions for a repeated line, *viz.*, $A_{ik} = 0$ amount to only three independent conditions.

Dually a line conic may degenerate into two distinct points with four independent constants or a twice counted point which has but two.

There is however a third variety of degenerate conic which can be obtained by a limit process. In general the line conic derived from two distinct lines is their point of intersection counted twice, while a line conic arising from a repeated line is arbitrary. But if we think of two lines as approaching coincidence by rotating about their common point which remains fixed the line equation will no longer vanish identically but will represent a definite point (twice counted), namely the original point of intersection. Or

[1] We should obtain the same result from the fundamental theorem for the polars of any two points on the line pass through y which is therefore the pole of the line.

we may imagine two points to approach coincidence in such a way that their junction remains a determinate line. We are thus led to a sort of hybrid conic made up of a line and a point which are incident. The line repeated is a point conic of which the point taken twice is the associated line conic and *vice versa*.[1]

To find the equation of such a conic consisting say of a repeated line v and a repeated point y which are incident we start with the general point and line conic f and ϕ

$$\begin{aligned} f\colon\ & a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{23}x_2x_3 \\ & \qquad + 2a_{31}x_3x_1 + 2a_{12}x_1x_2 = 0. \\ \phi\colon\ & b_{11}u_1^2 + b_{22}u_2^2 + b_{33}u_3^2 + 2b_{23}u_2u_3 \\ & \qquad + 2b_{31}u_3u_1 + 2b_{12}u_1u_2 = 0. \end{aligned} \tag{1}$$

and stipulate that f shall be the square of the line v and ϕ the square of the point y.

Then

$$f \equiv k(v_1x_1 + v_2x_2 + v_3x_3)^2$$

or

$$a_{11}:a_{22}:a_{33}:a_{23}:a_{31}:a_{12} = v_1^2:v_2^2:v_3^2:v_2v_3:v_3v_1:v_1v_2.$$

[1] We encountered such a conic in Part I of this chapter (§132). We said there that when the quadratics in the parametric equations belong to an involution the conic reduces to a repeated line. But we were then speaking of the conic as a point locus. We should now say that such a conic degenerates to a point pair since it still contains four constants and the line equation does not vanish identically. In fact if the conic is written

$$x_i = a_it^2 + c_i$$

as is always possible when the quadratics belong to an involution, the line equation is

$$(au)\ (cu) = 0$$

which obviously represents a pair of points.

Again when the quadratics have a common factor the equations may be written

$$x_i = (t - t_1)(a_it + b_i).$$

The conic now depends on three constants (the three ratios a_i/b_i and t_1 diminished by one for the relation satisfied by the x's) and is the type here considered. The line equation now is

$$\{(au)t_1 - (bu)\}^2 + 4(au)t_1\ (bu) \equiv \{(au)t_1 + (bu)\}^2 = 0$$

which represents the square of a point, namely the point t_1 on the line $x_i = a_it + b_i$. Two of the constants thus determine the line while the third t_1 fixes the point on the line.

Whence we obtain the relations

$$a_{11}:a_{12}:a_{13} = a_{21}:a_{22}:a_{23} = a_{31}:a_{32}:a_{33} = v_1:v_2:v_3. \tag{2}$$

Similarly

$$b_{11}:b_{12}:b_{13} = b_{21}:b_{22}:b_{23} = b_{31}:b_{32}:b_{33} = y_1:y_2:y_3. \tag{3}$$

The conic then is represented by the two equations (1) which are restricted by the conditions (2), (3) and

$$v_1y_1 + v_2y_2 + v_3y_3 = 0 \tag{4}$$

since y and v are incident.

We have thus four essentially distinct varieties of conics which are listed in the following table:

Point Conic.	*Line Conic.*
1°. Proper conic: $f = 0$, $A \neq 0$.	Proper conic: $\phi = 0$, $B \neq 0$.
2°. Line pair. $A = 0$, not every $A_{ik} = 0$. As a line conic, a repeated point.	Point pair. $B = 0$, not every $B_{ik} = 0$. As a point conic a repeated line.

3°. A double point and a double line in united position. Conditions (2), (3), (4) above.

4°. Repeated line. Every $A_{ik} = 0$. Line equation arbitrary.	Repeated point. Every $B_{ik} = 0$. Point equation arbitrary.

EXERCISES

1. Expand the discriminant A of the general conic §137.

2. Write the discriminant in determinant and expanded form of

$$ax^2 + 2hxy + by^2 + 2fy + 2gx + c = 0.$$

3. Write the discriminant of the circle

$$k(x^2 + y^2) + 2gx + 2fy + c = 0.$$

4. Prove that when a conic has a double point it actually breaks up into a pair of lines. (Show that a line joining a point on the curve to the double point lies wholly on the curve.)

5. The discriminant of a conic is also the Hessian (§101).

6. A double point of a conic may be defined as a center that lies on the curve, *i. e.*, the condition that the center lie on the curve is $A = 0$.

7. Verify that the line equations of the line pairs $xy = 0$, $x^2 + y^2 = 0$ represent repeated points. What is the point equation of the Absolute (the circular points considered as a degenerate line conic)?

8. Replace the constant term in each of the conics in Ex. 12, §136 by k. Then find the value of k which will make each conic degenerate.

9. Show that the following conics have double points. Find the coördinates of the double points and the equations of the lines of which the conics consist.

$$\begin{aligned} 6x^2 - 4y^2 - 63z^2 + 43yz - 13zx - 5xy &= 0 \\ 6x^2 + 3y^2 + z^2 + 2yz - 4zx - 8xy &= 0 \\ 5x^2 + 6y^2 + 15z^2 - 24yz + 18zx - 12xy &= 0. \end{aligned}$$

10. Show that the following line conics degenerate into point pairs and find the equations of the lines on which the point pairs lie:

$$\begin{aligned} 24u_1^2 - 15u_2^2 - 18u_3^2 + 37u_2u_3 - 24u_3u_1 + 2u_1u_2 &= 0 \\ 27u_1^2 - 5u_2^2 + 19u_3^2 - 2u_2u_3 - 54u_3u_1 + 18u_1u_2 &= 0 \\ u_1^2 + 12u_2^2 + 25u_3^2 - 20u_2u_3 - 10u_3u_1 + 4u_1u_2 &= 0. \end{aligned}$$

11. Find the polar of the point (y_1, y_2, y_3) with respect to the pair of lines $(ux)(vx) \equiv (u_1x_1 + u_2x_2 + u_3x_3)(v_1x_1 + v_2x_2 + v_3x_3) = 0$, also the pole of the line (v_1, v_2, v_3) with respect to the pair of points $(xu)(yv) = 0$ and identify the results with those of §57.

12. Extend the results of §§137–8 to the quadric surface. In particular define double point of a quadric, find the discriminant (condition for a double point, *i. e.*, that the quadric be a cone), find coördinates of the double point and show that the plane equation of a cone is the double point repeated. Investigate the conditions (in terms of minors of the discriminant) that a quadric be a pair of distinct planes, a repeated plane.

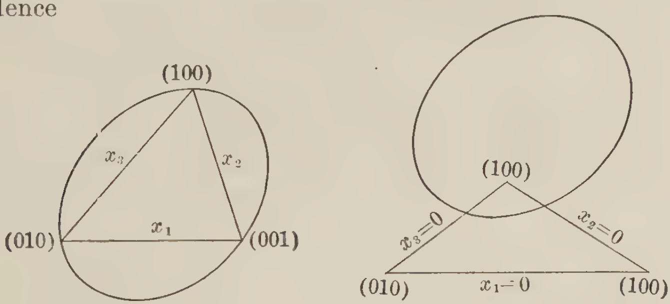
13. The discriminant of an n-ary quadratic is a symmetrical determinant, the Hessian of the form.

140. Forms of conics referred to various triangles.—The form of the equation of a conic depends of course upon the relation of the curve to the triangle of reference.

Take as the general conic in points

$$f\colon a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{23}x_2x_3 + 2a_{31}x_3x_1 + 2a_{12}x_1x_2 = 0. \quad (1)$$

and let us consider how this is modified by a special placing of the reference triangle. For example if f is on the vertex $u_1 = 0$ we must have, substituting (1, 0, 0) in (1), $a_{11} = 0$. Hence



1°. *The equation of a conic circumscribing the triangle of reference may be written in the form*

$$a_{23}x_2x_3 + a_{31}x_3x_1 + a_{12}x_1x_2 = 0. \tag{2}$$

If the a's are regarded as parameters (2) represents the ∞^2 conics on the vertices of the reference triangle.

Again if the side $x_1 = 0$ is the polar of the vertex $u_1 = 0$ we have, equating the polar of $(1, 0, 0)$ to a multiple of x_1

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \equiv a_{11}x_1$$

whence $a_{12} = a_{13} = 0$. Therefore

2°. *The equation of a conic referred to a self-polar triangle takes the form*

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 = 0. \tag{3}$$

Thus when the triangle of reference is self-polar the equation of a conic contains only square terms while if the triangle of reference is inscribed the equation contains only product terms. It is clear from equation (3) that there is a double infinity or *net* of conics with a common self-polar triangle.

Suppose now that the conic is inscribed in the reference triangle. If the conic touches the side $x_1 = 0$, *i. e.*, if the

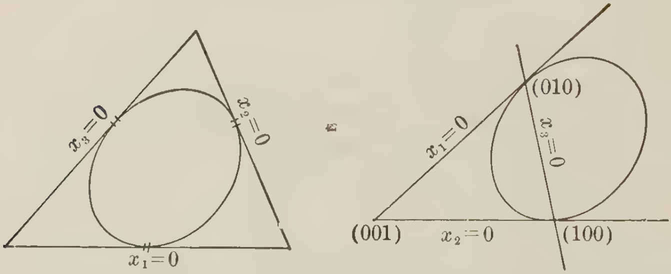
line cuts the curve in coincident points then the result of combining $x_1 = 0$ with (1), *viz.*,

$$a_{22}x_2^2 + 2a_{23}x_2x_3 + a_{33}x_3^2 = 0$$

must be a square. The condition is $a_{23}^2 = a_{22}a_{33}$ and we may say

3°. *The equation of a conic which is inscribed in the triangle of reference assumes the form*

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2\sqrt{a_{22}a_{33}}x_2x_3 + 2\sqrt{a_{33}a_{11}}x_3x_1 + 2\sqrt{a_{11}a_{22}}x_1x_2 = 0. \quad (4)$$



4°. *If the triangle of reference consist of two tangents and their chord of contact the equation of a conic may be reduced to the form*

$$x_1x_2 + a_{33}x_3^2 = 0. \quad (5)$$

For if $x_1 = 0$, $x_2 = 0$ are tangents and $x_3 = 0$ is their chord of contact, then the polars of (1, 0, 0), (0, 1, 0) and (0, 0, 1) can differ only by a constant from $x_2 = 0$, $x_1 = 0$ and $x_3 = 0$ respectively. So we may write

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &\equiv x_2 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &\equiv x_1 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &\equiv a_{33}x_3 \end{aligned} \quad (6)$$

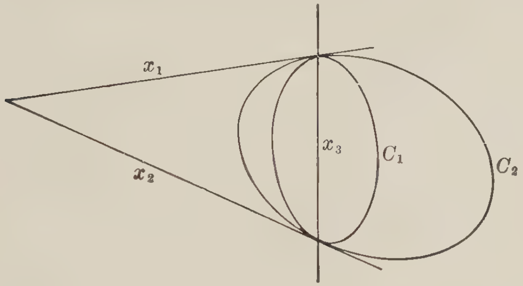
whence equating coefficients

$$a_{12} = a_{21} = 1, \; a_{33} = a_{33}$$

while all other a's are zero. Q. E. D.

Equation (5) which is the simplest form to which a proper conic can be reduced is the *canonical* equation.[1] If a_{33} is a parameter the equation represents a pencil of conics all touching two sides of the reference triangle at the same points and therefore having double contact with each other, $x_3 = 0$ being the common chord.

Since the general conic may, by suitably choosing the reference triangle, be made to assume any one of the several forms of this section, in investigating the properties of



conics it will be legitimate to use that form of the equation which is most convenient. For example consider two members of the pencil (5) say $C_1 \equiv k_1x_1x_2 + x_3^2$ and $C_2 \equiv k_2x_1x_2 + x_3^2$. We have at once

$$k_2C_1 - k_1C_2 \equiv (k_2 - k_1)x_3^2,$$

a relation which must hold when C_1 and C_2 are any two double-contact conics and x_3 their chord of contact. Therefore

If two conics have double contact their equations can be combined linearly to form the square of a line, their common chord, and conversely.

[1] *Cf.* §124. Also recall the standard equations of the parabola $y^2 = 4ax$, and the equilateral hyperbola $xy = a$ which illustrate the theorem. a_{33} may of course be removed from (5) by setting $x_3 = x_3'/\sqrt{a_{33}}$ or by taking $a_{33} = 1$ on the right in (6).

EXERCISES

1. Write the line equation and the discriminant of the several conics of this section.

2. Write line conics corresponding to the conics of this section and explain the relation of the reference triangle to each type.

3. Any proper conic can be reduced to the form (*a*) $x^2 + y^2 + z^2 = 0$, (*b*) $\sqrt{x} + \sqrt{y} + \sqrt{z} = 0$. Compare (3) and (4) above.

4. Canonical forms of degenerate conics are (*a*) of line pair $xy = 0$, (*b*) of an incident point and line $u_1^2 = 0$, $x_2^2 = 0$, (*c*) of a repeated line $x^2 = 0$.

5. Two conics (which meet in four distinct points) can be reduced to the forms $x^2 + y^2 + z^2 = 0$ and $ax^2 + by^2 + cz^2 = 0$, where $a + b + c = 0$. (Take as reference triangle the common self-polar triangle and as unit point some point on the second conic.)

6. Two conics (which meet in four distinct points) can be written in the forms $a_1yz + b_1zx + c_1xy = 0$, $a_2yz + b_2zx + c_2xy = 0$, where $a_1 + b_1 + c_1 = 0 = a_2 + b_2 + c_2$.

7. Show directly that a triangle inscribed in a conic and its polar triangle (triangle touching at vertices) are perspective.

8. The first polar of y with respect to a ternary form f is defined to be $\left(y \frac{\partial}{\partial x}\right)f \equiv \left(y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} + y_3 \frac{\partial}{\partial x_3}\right)f$ and the second polar of y is $\left(y \frac{\partial}{\partial x}\right)^2 f$, etc. Then if f is the triangle of reference considered as a cubic curve, *i. e.*, $f \equiv x_1x_2x_3 = 0$, show that $\left(y \frac{\partial}{\partial x}\right)f(x) = \left(x \frac{\partial}{\partial y}\right)^2 f(y)$. The first polar of y with respect to the reference triangle is a conic on the vertices and the second polar is the polar line of y as to the triangle (Ex. 7, §28). The second polar is also the polar of y with respect to the first polar (conic polar).

9. If C_1 and C_2 are any two conics then there are three members of the pencil $C_1 + \lambda C_2 = 0$ which consist of line pairs. For the discriminant of the pencil involves λ to the third degree. Account for these three degenerate members in a pencil of double-contact conics. Also discuss the degenerate members when C_1 and C_2 intersect in (*a*) two distinct, two coincident points, (*b*) three coincident and one additional point, (*c*) four coincident points. In which case does the cubic in λ have two equal, three equal roots?

10. If $\alpha = 0$, $\beta = 0$ are a pair of tangents to a conic C and $\gamma = 0$ is their chord of contact, then it is possible to determine k so that the conic can be written

$$1°: \quad C \equiv \alpha\beta + k\gamma^2.$$

For taking α, β, γ as triangle of reference, C must be a member of the pencil of double contact conics $\alpha\beta + k\gamma^2 = 0$ of which $\alpha\beta = 0$ and $\gamma^2 = 0$ are degenerate members. The identity can be written in the alternative forms

$$2°. \quad \gamma^2 \equiv C + k\alpha\beta \qquad\qquad 3°. \quad \alpha\beta \equiv \gamma^2 + kC.$$

Dualize.

11. The equation of a conic differs only by an additive constant from the equation of its asymptotes. For if $\gamma = 0$ (Ex. 10) be the line at infinity we may write (from 1°) $C \equiv \alpha\beta + kz^2$ where α and β are the asymptotes, thence setting $z = 1$ we have $C \equiv \alpha\beta + k$.

12. If $C \equiv ax^2 + 2hxy + by^2 + 2fy + 2gx + c$, the equation of the asymptotes is $C - \dfrac{\Delta}{ab - h^2} = 0$, where Δ is the discriminant of C. For by Ex. 11 the asymptotes $\equiv C + k$, hence it is only necessary to determine k so that $C + k = 0$ will break up into a pair of lines.

13. Find by the formula of Ex. 12 the asymptotes of the conics in Ex. 12, §136. Also find the asymptotes of the conics

$$\begin{aligned} 2x^2 + 5xy - 3y^2 + 7x - 11y - 9 &= 0 \\ 13x^2 - 2xy + 6y^2 - 17x - 10y + 24 &= 0 \\ 14x^2 + 25xy + 8y^2 - 15x + 20y + 18 &= 0. \end{aligned}$$

14. The asymptotes of the conic in Ex. 12 are parallel to the lines $ax^2 + 2hxy + by^2 = 0$.

15. By Ex. 10 find the equation of the chord of contact of a pair of tangents from a point on the directrix of a conic and show that it passes through the focus. Take the conic in the form

$$(1 - e^2)x^2 + y^2 - 2px + p^2 = 0$$

when the y-axis is the directrix and the focus is the point $(p, 0)$.

16. Show that the two pairs of conics have double contact and find (a) the chord of contact, (b) the pole of the chord of contact, (c) the pair of common tangents, (d) the two contacts

$$\begin{cases} 6x_1^2 + 10x_2^2 + 7x_3^2 + 5x_2x_3 + 21x_3x_1 + 3x_1x_2 = 0 \\ 11x_1^2 + 18x_2^2 + 8x_3^2 + 25x_2x_3 + 39x_3x_1 + 15x_1x_2 = 0 \end{cases}$$

$$\begin{cases} 3u_1^2 + 12u_2^2 + 48u_3^2 - 11u_2u_3 - 58u_3u_1 - 35u_1u_2 = 0 \\ 11u_1^2 - 20u_2^2 + 129u_2u_3 - 66u_3u_1 + 25u_1u_2 = 0. \end{cases}$$

17. If two conics C_1, C_2 have double contact with a third conic S, their chords of contacts with S and one pair of their common chords meet in a point and form a harmonic pencil. For we have $S + k_1C_1 \equiv a^2u^2$ and $S + k_2C_2 \equiv b^2v^2$ where u and v are the chords of

contact. Thence subtracting, $k_1C_1 - k_2C_2 \equiv a^2u^2 - b^2v^2$, which says that the function on the right represents a degenerate member of the pencil determined by C_1 and C_2. But it is also a pair of lines of the (line) pencil determined by u and v. Why are the lines harmonic with u and v?

18. The chords of contact of two conics with their common tangents meet at the intersection of a pair of their common chords. (Cor. of Ex. 17 when S breaks up into a pair of lines.) Prove Ex. 6 (dual), §74 as a corollary of Ex. 17 by supposing C_1 and C_2 to reduce to tangent pairs.

141. Polar of one conic with respect to another.—Consider two conics f and ϕ whose point equations are

$$f: a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{23}x_2x_3 + 2a_{31}x_3x_1 + 2a_{12}x_1x_2 = 0 \quad (1)$$

$$\phi: b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + 2b_{23}x_2x_3 + 2b_{31}x_3x_1 + 2b_{12}x_1x_2 = 0 \quad (2)$$

and whose line equations denoted by F and Φ respectively are

$$F: A_{11}u_1^2 + A_{22}u_2^2 + A_{33}u_3^2 + 2A_{23}u_2u_3 + 2A_{31}u_3u_1 + 2A_{12}u_1u_2 = 0 \quad (3)$$

$$\Phi: B_{11}u_1^2 + B_{22}u_2^2 + B_{33}u_3^2 + 2B_{23}u_2u_3 + 2B_{31}u_3u_1 + 2B_{12}u_1u_2 = 0. \quad (4)$$

If we substitute differential symbols in Φ, replacing u_i by $\partial/\partial x_i$, and operate on f we shall obtain a function of the coefficients of the conics which is called the *polar of* Φ *with respect to* f.[1] Neglecting a numerical factor the polar is

$$\Theta': a_{11}B_{11} + a_{22}B_{22} + a_{33}B_{33} + 2a_{23}B_{23} + 2a_{31}B_{31} + 2a_{12}B_{12}. \quad (5)$$

[1] This is a natural extension of the method for binary forms. For the relation connecting the coördinates x of the points of a range with the coördinates u of the lines of a pencil is (§47, last theorem) $u_1x_1 + u_2x_2 = 0$, or $x_1/x_2 = u_2/-u_1$. So that the "line equation" of a binary form is derived from the point equation by substituting for x_1, x_2, respectively the line coördinates u_2, $-u_1$. Then considering the u's as differential symbols ($u_i = \partial/\partial x_i$) we have the operator for finding the polar of one form with respect to another. But in the binary domain either form may be taken in line coördinates and the apolarity condition is unique.

It is easy to see that we should get the same result by substituting $\partial/\partial u_i$ for x_i in f and operating on Φ. It would seem therefore that we might with equal weight call (5) the polar of f with respect to Φ as in the case of binary forms. But here it makes a radical difference if f is taken in lines and Φ in points. Accordingly we shall understand that the "polar of Φ with respect to f" always refers to the function derived from the line equation of Φ and the point equation of f.[1]

Thus to find the polar of f with respect to ϕ we operate with the line equation F of f on the point equation ϕ obtaining

$$\Theta: A_{11}b_{11} + A_{22}b_{22} + A_{33}b_{33} + 2A_{23}b_{23} + 2A_{31}b_{31} + 2A_{12}b_{12}. \quad (6)$$

When either polar vanishes the two conics are said to be *apolar*, a term which is therefore ambiguous when applied to two conics. But the distinction between the two polars can be maintained by the convention adopted above.

Generally the polar of a curve φ^m of class m with respect to a curve f^n of order n, $m \leq n$, is the function obtained by operating on f in points with the line equation of φ. And *the curves are apolar when the polar of one with respect to the other vanishes identically.*

142. Geometrical relation of apolar conics.—Let the conic f be referred to a self-polar triangle when it can be written in the form (§140)

$$f: a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 = 0. \quad (1)$$

Then since $a_{12} = a_{13} = a_{23} = 0$ the polar of f with respect to ϕ is ((6) §141)

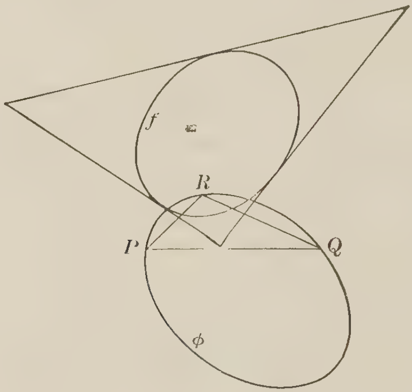
$$\Theta: A_{11}b_{11} + A_{22}b_{22} + A_{33}b_{33} = 0. \quad (2)$$

[1] Whether it is calculated by operating on the point equation of f with the line equation of Φ or by operating on the line equation of Φ with the point equation of f is of course wholly immaterial,—the essential thing is that Φ be written in lines and f in points.

If now the triangle of reference be inscribed in ϕ we must have ((2) §140) $b_{11} = b_{22} = b_{33} = 0$, *i. e.*, $\Theta = 0$ is a necessary condition that there exist a triangle[1] inscribed in ϕ and self-polar to f.

Next we shall prove that it is but one condition for the conics to be so related and that therefore $\Theta = 0$ is *the* condition.

Let the polar as to f of any point P of ϕ cut ϕ in Q and R. Then the polar of Q with respect to f must pass



through P. And it will pass through R if and only if the coördinates of R satisfy its equation which imposes a single condition on the coefficients of the two conics. This condition fulfilled, the polar of R contains both P and Q and the triangle PQR is inscribed in ϕ and self-polar to f. Q. E. D. Hence

$\Theta = 0$ *is the necessary and sufficient condition that it be possible to inscribe in* ϕ *a triangle which is self-polar to* f.

Again if the triangle of reference is self-polar with respect

[1] Here the reference triangle.

to ϕ we have $b_{12} = b_{13} = b_{23} = 0$ and the equation of ϕ reduces to

$$\phi\colon b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 = 0 \tag{3}$$

and Θ becomes

$$(a_{22}a_{33} - a_{23}^2)b_{11} + (a_{33}a_{11} - a_{31}^2)b_{22} + (a_{11}a_{22} - a_{12}^2)b_{33}. \tag{4}$$

If now the reference triangle is circumscribed to f, $a_{22}a_{33} = a_{23}^2$, $a_{33}a_{11} = a_{31}^2$, $a_{11}a_{22} = a_{12}^2$ (§140, (4)) and $\Theta = 0$. As before it is a single condition on the conics for a triangle which is circumscribed to f to be self-polar with respect to ϕ. Therefore $\Theta = 0$ *is also the necessary and sufficient condition that a triangle can be drawn at once circumscribed to* f *and self-polar with respect to* ϕ.

In a similar way it is proved that

The necessary and sufficient condition that a triangle can be constructed (a) *inscribed in* f *and self-polar to* ϕ *or* (b) *circumscribed to* ϕ *and self-polar to* f *is that the polar of* ϕ *with respect to* f *shall vanish, i. e.,* $\Theta' = 0$.

In fact when $\Theta = 0$ a single infinity of each class of characteristic triangles can be drawn[1] and a similar statement holds for Θ'.

Since the vanishing of Θ or Θ' implies a projective relation between the two conics these two polars are (simultaneous) invariants of the conics. Together with the two discriminants they constitute a complete system of pure invariants.

143. Metrical polar properties.—We shall first establish the theorem:

A directrix of a conic is the polar of a focus.

The equation of a central conic in homogeneous Cartesian coördinates may be written

$$b^2x^2 \pm a^2y^2 - a^2b^2z^2 = 0 \tag{1}$$

[1] Such a property, *i. e.*, one which if true once is true an infinite number of times is called *poristic*.

of which the coördinates of the real foci are $(\pm ae, 0, 1)$. The polar lines of these points are

$$\pm aeb^2x - a^2b^2z = 0,$$

whence returning to the non-homogeneous form by writing $z = 1$,

$$x = \pm a/e \tag{2}$$

which are the equations of the real directrices.

The polars of the two imaginary foci are naturally defined to be (imaginary) directrices.

When the parabola is written

$$y^2 - 4azx = 0 \tag{3}$$

the coördinates of the focus are $(a, 0, 1)$.
The polar of the focus is accordingly

$$-4a(az + x) = 0, \text{ or when } z = 1,\ x = -a, \tag{4}$$

the familiar equation of the directrix.

Again let the general conic in homogeneous Cartesian coördinates be written

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0. \tag{5}$$

The homogeneous metrical equation of the circular points is $u^2 + v^2 = 0$. Operating with this on (5) we find as the polar of the absolute with respect to the conic $2(a + b)$. This will vanish if $a = -b$, *i. e.*, the equation of a conic apolar to the absolute is

$$ax^2 - ay^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

Setting $z = 0$ we have $ax^2 + 2hxy - ay^2 = 0$ as the equation of the lines from $(0, 0, 1)$ to the intersection of the conic with $\mathfrak{L}$. Now these lines are perpendicular since the product of the roots of the equation considered as a quadratic in y/x is -1. It follows that the conic goes off to infinity in the direction of perpendicular lines, in other

words the conic is a rectangular hyperbola. Conversely if (5) is a rectangular hyperbola, the polar of the circular points with respect to the conic is zero. Hence

A necessary and sufficient condition that a conic be a rectangular hyperbola is that the polar of the absolute with respect to the conic vanish.

EXERCISES

In these exercises we use the notation of §141, denoting the two conics in points by f and ϕ, the polar of f with respect to ϕ by Θ and the polar of ϕ with respect to f by Θ'. Further the discriminants of the two conics are denoted by Δ and Δ' respectively.

1. Write Θ and Θ' in terms of the coefficients of the point equations of the two conics.

2. Show that Θ can be written

$$\left(b_{11}\frac{\partial}{\partial a_{11}} + b_{22}\frac{\partial}{\partial a_{22}} + b_{33}\frac{\partial}{\partial a_{33}} + b_{23}\frac{\partial}{\partial a_{23}} + b_{31}\frac{\partial}{\partial a_{31}} + b_{12}\frac{\partial}{\partial a_{12}}\right)\Delta$$

and write a similar expression for Θ'.

3. Write the discriminant of $kf + \phi$ and show that the expanded form is

$$\Delta k^3 + \Theta k^2 + \Theta' k + \Delta'.$$

The vanishing of this cubic is the condition that a conic of the pencil degenerate to a pair of lines.

4. Write the cubic (Ex. 3) for the pairs of conics

$$(1)\begin{array}{l} f: x^2 + y^2 + z^2 = 0 \\ \phi: ax^2 + by^2 + cz^2 = 0 \end{array} \qquad (2)\begin{array}{l} f: yz + zx + xy = 0 \\ \phi: fyz + gzx + hxy = 0 \end{array}$$

$$(3)\begin{array}{l} f: x^2 + y^2 + z^2 = 0 \\ \phi: ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \end{array}$$

$$(4)\begin{array}{l} f: yz + zx + xy = 0 \\ \phi: ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0. \end{array}$$

5. If the cubic (Ex. 3) have two equal roots the conics have contact. Find the condition in terms of the invariants (§114).

6. Find the invariant condition that the two conics *osculate* (meet in three coincident points). See Ex. 9(*b*), §140 and Ex. 20, §115.

7. If k is eliminated between the cubic (Ex. 3) and $kf + \phi = 0$ the resulting function set equal to zero is the ternary equation of the three degenerate conics of the pencil. Write this equation.

8. Find the invariant relation on two conics when a triangle can be inscribed in one and circumscribed to the other. (Take the triangle as reference triangle, write the specialized form of the two conics, calculate the invariants.) *Ans.* $\Theta^2 - 4\Delta\Theta' = 0$.

9. The polar of a conic with respect to a self-polar triangle considered as a cubic curve vanishes identically.

10. (*a*) If the polar of a line pair with respect to a conic vanishes the lines intersect on the conic. (*b*) But if the polar of a conic with respect to a line pair vanishes, the lines are conjugate lines of the conic. Let $f \equiv a_{11}x_1^2 + \ . \ . \ .$ and $\phi \equiv x_1x_2 = 0$ and apply Ex. 23, §136.

N.B. In (*a*) ϕ must be written in lines and in (*b*) f must be taken in lines.

11. If the polar of a point pair with respect to a conic vanishes, the points are conjugate points of the conic. And if the polar of the conic with respect to the point pair vanishes, the junction of the points touches the conic.

12. When applied to a pair of points and a pair of lines, apolar and harmonic are equivalent terms.

13. The k-cubic for the conics in Ex. 10 reduces to a quadratic multiplied by k. What is the geometric implication of the vanishing of the discriminant of this quadratic? Compare Ex. 8. Dualize and write the corresponding invariant conditions.

14. Prove the last theorem of §143 as an application of Ex. 11

15. The necessary and sufficient condition that a conic be a parabola is that the polar of the conic with respect to the absolute vanish.

16. If two triangles T_1 and T_2 are self-polar with respect to a conic f, their vertices lie on a second conic ϕ. Take $f \equiv x_1^2 + x_2^2 + x_3^2 = 0$ choosing T_2 as triangle of reference. Then let ϕ pass through the vertices of T_1 and two vertices of the reference triangle. We have thus a triangle T_1 inscribed in ϕ and self-polar to f. Show that under the hypothesis ϕ must pass through the third vertex of the triangle of reference.

17. Prove the statement §142 that when two conics are apolar, ∞^1 triangles can be drawn inscribed in one and self-polar to the other (or the dual). Write the conics as is possible $f \equiv ax^2 + by^2 + cz^2 = 0$, $\phi \equiv yx + zx + xy = 0$. And let the polar with respect to f of any point P on ϕ cut ϕ in points Q and R. Then show that the polar of R with respect to f passes through Q as well as P. How does this prove the theorem? Or use the conics K and U, §134.

18. If three conics have a common self-polar triangle, the Jacobian

of the three breaks up into the three lines themselves. (Take the common self-polar triangle as triangle of reference.)

19. If each of three conics is apolar to the Jacobian of the three in points then either the conics are identical or they have a common self-polar triangle. (Take as two conics $x^2 + y^2 + z^2 = 0$, $ax^2 + by^2 + cz^2 = 0$.)

20. If a curve ϕ is apolar to a set of curves $f_1, f_2, \ldots f_r$, ϕ is apolar to any linear combination of the f's.

21. There are $6 - r$ linearly independent conics apolar to r linearly independent conics. In particular there is a unique conic apolar to five linearly independent conics.

22. If a conic is apolar to all first (conic) polars of a ternary cubic f it is apolar to the cubic. (All first polars are linear combinations of $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}$.)

23. There are three linearly independent conics apolar to a ternary cubic.

24. It is one condition on a ternary quartic to have a given conic as an apolar conic. Find this condition as a symmetrical 6-row determinant.

25. A conic is apolar to the square (or any higher power) of any tangent line.

26. If a ternary quartic can be written as a sum of five fourth powers $X_1^4 + X_2^4 + X_3^4 + X_4^4 + X_5^4$, it must possess an apolar conic. For a conic can be inscribed in the five lines represented by $X_i = 0$. Apply Exs. 20 and 25.

27. As in §134 find a correspondence between line conics and binary quartics on the fundamental conic K. (Ask that the polar of the line conic say $Au_1^2 + Bu_2^2 + Cu_3^2 + 2Fu_2u_3 + 2Gu_3u_1 + 2Hu_1u_2 = 0$ with respect to K vanish. The condition is $B = 4G$. The conic then corresponds to the binary quartic say ϕ giving the common lines of the conic and K.) Prove that if this line conic is apolar to the point conic U, §134, the binary quartics corresponding to the two conics are apolar.

28. If two conics C_1 and C_2 are each apolar to a third conic K and if C_1 cuts K in four points f, while C_2 touches the four tangents to K at the points f, express as an invariant condition on f the condition that the polar of C_2 with respect to C_1 vanish.

29. Find three linearly independent line conics apolar to the ternary cubic $x^3 + y^3 + z^3 + 6axyz = 0$ and write the Jacobian of the net determined by the conics. (See Ex. 22 and example in §99 under 3°.)

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Ans. $vw - au^2 = 0$, $wu - av^2 = 0$, $uv - aw^2 = 0$. $a(u^3 + v^3 + w^3) + (1 - 4a^3)uvw = 0$, the *Cayleyan* of the cubic.

30. As in Ex. 29 find the Cayleyan of the cubic $x^3 + y^3 - 3xyz = 0$.

31. The Hessian and the Cayleyan of a ternary cubic are the respective Jacobians of apolar nets of conics. (The Hessian is the Jacobian of first polars.)

144. Reflexion of a conic into itself.—When the conic is written in the canonical form

$$x_1 = t^2, \qquad x_2 = 2t, \qquad x_3 = 1, \tag{1}$$

or

$$x_2^2 - 4x_3x_1 = 0 \tag{2}$$

we saw (§130) that lines of a pencil cut the conic in pairs of points in a quadratic involution. For simplicity consider the involution set up by the vertex $u_2 = 0$. Conjugate pairs in the involution then satisfy the relation

$$T: t' + t = 0 \qquad \text{or} \qquad t' = -t \tag{3}$$

which may be regarded as the *transformation* (alibi) which sends the point t into the point t'. But there is a ternary transformation associated with the binary. For the parameters t and t' are attached to two points x and x' which (as ternary elements) are harmonically separated from the vertex u_2 by its polar line x_2. The relation between the points x and x', found by replacing t in (1) by $-t'$ from (3) is

$$R: x_1' = x_1, \qquad x_2' = -x_2, \qquad x_3' = x_3. \tag{4}$$

Thus the binary transformation T generates the ternary transformation R which carries the point x into the point x'. Applied a second time the transformation carries x' back into x. Hence R is of period two and is called an *involutory transformation* or a *reflexion*.[1]

It appears then that any point and its polar line determine

[1] Also called a *harmonic perspectivity*. It is sometimes called an involution but we shall restrict involution to the binary domain.

a reflexion which transforms a conic into itself since (4) manifestly leaves the equation (2) of the conic unchanged. The point is called the *center* and the polar line the *axis* of the reflexion. x and x' are *conjugate* points under the reflexion and either is the reflexion or the *image* of the other in the axis from the center. Or we say that

A conic may be reflected into itself with any point as center and the polar line of the point as axis.

It is instructive to compare the binary with the ternary transformation. Both are of period two. Conjugate elements t of the involution name conjugate points x of the reflexion. It follows that fixed points of the involution are also fixed points of the reflexion. There are however conjugate points of R (not lying on the conic) which are not elements of T. While T has only two fixed (double points) any point on the axis is a fixed point of R since if x lies on the axis, $x' \equiv x$ (§43, 1°). Again the image of any point of a line on the center is another point of the same line, *i. e.*, R merely shifts the points of such a line among themselves. Therefore

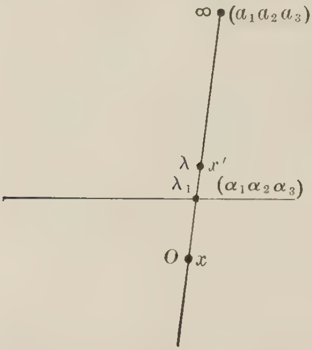
The axis of reflexion is a line of fixed points and the center is a point of fixed lines.

A reflexion may of course be considered quite apart from the conic. Thus any point a and any line α as center and axis determine a reflexion in the plane of which conjugate points are pairs x and x' collinear with a and harmonically separated from a by the axis. The equations of the transformation are simply the relations which connect the coördinates of the points x and x'. When the center and axis are taken as a vertex and side of the reference triangle the reflexion assumes the canonical form (4). The general form can be written down from (4) however. For $x_1 = 0$, $x_3 = 0$ are merely two lines on the center while $x_2 = 0$ is the axis. Consequently if $(wx) \equiv w_1x_1 + w_2x_2 + w_3x_3 = 0$ is the

equation of the axis and $(ux) = 0$, $(vx) = 0$ are two lines through the center the equations of the reflexion are

$$(ux') = (ux), \qquad (vx') = (vx), \qquad (wx') = -(wx). \tag{5}$$

Another form of the transformation can be found as follows. Let (a_1, a_2, a_3) and $(\alpha_1, \alpha_2, \alpha_3)$ respectively be the coördinates of the center a and axis α while (x_1, x_2, x_3) and (x_1', x_2', x_3') are the coördinates of a pair of conjugate points. Take a and x (parameters ∞, 0) as base points in the parametric representation of the line ax and assign to x' the parameter λ and to the intersection of the line ax with α the parameter λ_1. Then (§55)



$$x_i' = x_i + \lambda a_i, \quad i = 1, 2, 3. \tag{6}$$

If x' is on the axis

$$(\alpha x) \equiv \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$$

we must have by (6)

$$(\alpha x) + \lambda_1(\alpha a) = 0, \text{ whence}$$
$$\lambda_1 = -(\alpha x)/(\alpha a).$$

But by the harmonic relation $(0\lambda \mid \lambda_1 \infty) = -1$, *i. e.*, $\lambda = 2\lambda_1$ and the equations of the reflexion are

$$x_i' = (\alpha a)x_i - 2(\alpha x)a_i. \tag{7}$$

A metrically canonical form of a reflexion is obtained by taking the center at infinity in the direction of the perpendicular to the axis.[1] Conjugate points are then symmetrical with respect to the axis and the rectangular equations of the reflexion may be written

$$x' = x, \qquad\qquad y' = -y. \tag{8}$$

[1] The center is then the *zenith* of the line.

The axis is now a line of symmetry of any curve left invariant by the reflexion.[1]

Again if $\mathcal{L}$ is taken as the axis, the origin is the center and the equations of the reflexion are

$$x' = -x, \qquad y' = -y.$$

Any invariant curve is said to be reflected in the origin which is a point of symmetry, *i. e.*, a *center* of the curve.

145. Polarity and the principle of duality.—If

$$a_{11}z_1^2 + a_{22}z_2^2 + a_{33}z_3^2 + 2a_{23}z_2z_3 + 2a_{31}z_3z_1 + 2a_{12}z_1z_2 = 0 \quad (1)$$

be a proper conic then we saw (§136) that the coördinates (u_1, u_2, u_3) of the polar line of a point (x_1, x_2, x_3) are given by the equations

$$\Pi: \begin{aligned} \sigma u_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ \sigma u_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ \sigma u_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3. \end{aligned} \quad (2)$$

These equations are however susceptible of another interpretation. For we may regard (2) as the *transformation* that changes the point x into its polar line u. The equations are then said to define a *polarity* Π. Likewise a transformation Π' that carries a line of the plane into its polar point is called a polarity. *A polarity is thus a projective correspondence between the points and lines of the polar system of a conic,—sending every point of the plane into its polar line and every line into its polar point.*[2]

[1] If the axis is considered a mirror with the source of light at the zenith one half of the curve is the image of the other half in the ordinary physical reflexion. General reflexion is thus the abstract projective view of physical reflexion.

[2] It should be noted that a polarity is not the most general transformation which interchanges points with lines. For the coefficients in (2) and (3) in virtue of their connection with a conic are not independent, in fact $a_{ik} = a_{ki}$ and $A_{ik} = A_{ki}$. A polarity thus contains but five essential constants while the most general transformation in which the coefficients are all independent possesses eight.

Solving (2) we obtain

$$\Pi^{-1}:\quad \begin{aligned} \rho x_1 &= A_{11}u_1 + A_{12}u_2 + A_{13}u_3 \\ \rho x_2 &= A_{21}u_1 + A_{22}u_2 + A_{23}u_3 \\ \rho x_3 &= A_{31}u_1 + A_{32}u_2 + A_{33}u_3 \end{aligned} \tag{3}$$

the *inverse* of Π, a polarity which carries the line u back into its pole x.

It is clear from the foregoing discussion that (2) and (3) are the analytic formulas for transforming a curve into its polar reciprocal (§75) with respect to (1) as auxiliary conic. By (3) the polar reciprocal of the point curve $f(x_1, x_2, x_3) = 0$ is

$$f(A_{11}u_1 + A_{12}u_2 + A_{13}u_3,\ A_{21}u_1 + A_{22}u_2 + A_{23}u_3,\ A_{31}u_1 + A_{32}u_2 + A_{33}u_3) = 0.$$

And by (2) the reciprocal of a line curve $\phi(u_1, u_2, u_3) = 0$ is

$$\phi(a_{11}x_1 + a_{12}x_2 + a_{13}x_3, a_{21}x_1 + a_{22}x_2 + a_{23}x_3, a_{31}x_1 + a_{32}x_2 + a_{33}x_3) = 0.$$

Since polarities Π and Π^{-1} are linear in both sets of variables it is plain that the two equations f just written are of the same degree, so are the equations ϕ. Hence the order of a curve is equal to the class of its polar reciprocal and *vice versa*.

Suppose now that the conic be taken in the form[1]

$$z_1^2 + z_2^2 + z_3^2 = 0. \tag{4}$$

The polar of (x_1, x_2, x_3) is $x_1z_1 + x_2z_2 + x_3z_3 = 0$ so that the coördinates of the polar line are simply the coefficients in its equation, *i. e.*,

$$u_1 = x_1, \quad u_2 = x_2, \quad u_3 = x_3. \tag{5}$$

The polarity thus reduces to the dual transformation (5), consequently

[1] Which is always possible by referring the conic to a self polar triangle (§140 (3)) and then replacing x_i by $z_i/\sqrt{a_{ii}}$

The dual of a curve (or figure) is its polar reciprocal with respect to a conic of the form $x_1^2 + x_2^2 + x_3^2 = 0$.[1]

While the equation of this conic is real the conic is, for a real triangle of reference, wholly imaginary.

EXERCISES

1. Find the equations representing the reflexions which have the vertices and opposite sides of the triangle of reference as corresponding centers and axes. Show that all three transform into themselves the four points $(1, \pm 1, \pm 1)$.

2. Find the reflexions set up by the following points and lines as centers and axes

centers	axes
$(0, 1, 0)$	$x_1 + 2x_2 - 3x_3 = 0$
$(1, -1, 0)$	$x_1 - x_2 = 0$
$(1, 0, 0)$	$2x_1 + x_2 - 3x_3 = 0.$

Show that each has the invariant conic

$$4x_1^2 + 4x_2^2 + 9x_3^2 - 12x_2x_3 - 12x_3x_1 + 4x_1x_2 = 0.$$

3. Find the reflexions with the indicated centers that transform into themselves the following conics:

centers	conic
$(1, 0, 0), (0, 1, 0), (0, 0, 1)$	$ax_1^2 + bx_2^2 + cx_3^2 = 0$
$(1, 0, 0), (0, 1, 0), (0, 0, 1)$	$x_1^2 + x_2^2 + x_3^2 - 2x_2x_3 - 2x_3x_1 - 2x_1x_2 = 0$
$(p, 0, 1)$	$(1 - e^2)x^2 + y^2 - 2pzx + p^2z^2 = 0$
(a_1, a_2, a_3)	the general conic.

4. Find the polarities determined by the following points and conics

$(1, 1, 1)$	$2fyz + 2gzx + 2hxy = 0$
$(0, 0, 1)$	$x^2 + y^2 + a^2z^2 - 2ayz - 2azx = 0.$

Find the polarities set up by the points and conics in Ex. 3.

5. Every polarity transforms into itself the base conic (conic

[1] Since the dual of a curve $f(x_1, x_2, x_3) = 0$ may be taken as $f(u_1, u_2, u_3) = 0$. The general dual transformation is

$$D: u_i = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3, \quad i = 1, 2, 3.$$

The curve obtained by applying D to $f(u_1, u_2, u_3)$ is however projectively equivalent to $f(x_1, x_2, x_3)$. For the process is equivalent to applying the transformation (5) (inverse) and then linearly transforming the x's. In other words the result of D on f is a reciprocation in conic (4) followed by a projection since a linear transformation (of points into points or lines into lines) amounts to a projection. (See below, §146.)

whose polar system comprises the points and lines interchanged by the polarity).

6. Find the base conics of the following polarities

$$(a)\quad u_1 = x_2, \qquad u_2 = x_1, \qquad u_3 = 2x_3$$

$$(b)\quad \begin{aligned} u_1 &= 2x_1 + 3x_2 + 5x_3 \\ u_2 &= 3x_1 - x_2 - 4x_3 \\ u_3 &= 5x_1 - 4x_2 + 6x_3 \end{aligned} \qquad (c)\quad \begin{aligned} u_1 &= ax_1 + bx_2 + cx_3 \\ u_2 &= bx_1 + dx_2 + ex_3 \\ u_3 &= cx_1 + ex_2 + fx_3. \end{aligned}$$

7. Find the polar reciprocal of the general conic with respect to the circle $x^2 + y^2 = k^2$. Show that if the conic is a parabola its reciprocal passes through the origin.

8. Find the reciprocal of the circle $(x - a)^2 + y^2 = b^2$ with respect to the circle $x^2 + y^2 = k^2$. Show that the reciprocal is an ellipse, a parabola or a hyperbola according as $a < b$, $a = b$, $a > b$ and thus verify the theorem 1°, §77. Prove also theorem 3°, §77.

9. Prove analytically Exs. 13, 14, §78.

10. Find the reciprocal of the general conic with respect to the parabola $y^2 = 4ax$. Determine the condition that the reciprocal be a parabola and interpret the condition geometrically.

11. Show that $x_1^2 + x_2^2 + x_3^2 = 0$ is a proper conic.

12. Find the conditions that the general conic be its own polar reciprocal with respect to the conic $x_1^2 + x_2^2 + x_3^2 = 0$, *i. e.*, that the conic be *self-dual*. Thus find a self-dual conic.

13. Find the value of λ such that the conic $x_2^2 + \lambda x_3x_1 = 0$ shall be self-dual.

14. Find the polar reciprocal of the conic $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ with respect to $y^2 - zx = 0$. Find the conditions that the reciprocal be identical with the original, *i. e.*, that the conic be *autopolar*. Thence find a conic autopolar with respect to $y^2 - zx = 0$.

15. Any curve autopolar with respect to a conic is self-dual in the projective sense for the base conic can always be transformed into the form in Ex. 12.

16. If two conics have double contact and one is autopolar with respect to the other, the second is then autopolar with respect to the first. (Take the conics as $ax_1x_2 + bx_3^2 = 0$ and $\alpha x_1x_2 + \beta x_3^2 = 0$.) Thence find a pair of mutually autopolar conics.

17. Show that the pairs of conics are mutually autopolar: (1) Any hyperbola and its conjugate, *e. g.*, $x^2/a^2 - y^2/b^2 \pm 1 = 0$

$$\begin{aligned} x_1^2 + 21x_2^2 + 13x_3^2 - 24x_2x_3 - 8x_3x_1 - 7x_1x_2 &= 0 \\ 7x_1^2 - 19x_2^2 + 5x_3^2 + 12x_2x_3 - 16x_3x_1 + 15x_1x_2 &= 0. \end{aligned} \qquad (2)$$

CHAPTER XI

COLLINEATIONS IN THE PLANE

146. The notion of planar collineation.—Consider the linear transformation in two dimensions

$$T: \begin{aligned} \rho x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ \rho x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ \rho x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{aligned} \qquad (1)$$

where $a_{ik} \neq a_{ki}$ and $A \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0.$

We have seen (§62) that if x_i and x_i', $i = 1, 2, 3$, represent coördinates of the same point referred to two different triangles $x_i = 0$, $x_i' = 0$ then T effects a transformation of coördinates from one triangle to the other.[1] But exactly as in the binary domain (§49) the equations T are susceptible of another interpretation; for we may regard x_i and x_i' as coördinates of distinct points referred to the same triangle. T then instead of being the process which changes the coördinates of a point from one system to another becomes the operation that carries the point itself from one position to another.[2] From the latter point of view T is called a *collineation*.[3]

[1] That is, T is then an alias.

[2] The points x and x' might even lie in different planes in the most general interpretation of a linear transformation but for the sake of definiteness we shall restrict them to lie in the same plane.

[3] That is, an alibi.

Solving (1) for x_i we obtain

$$T^{-1}:\quad \begin{aligned} \rho' x_1 &= A_{11}x_1' + A_{21}x_2' + A_{31}x_3' \\ \rho' x_2 &= A_{12}x_1' + A_{22}x_2' + A_{32}x_3' \\ \rho' x_3 &= A_{13}x_1' + A_{23}x_2' + A_{33}x_3' \end{aligned} \qquad (2)$$

where A_{ik} is the cofactor of a_{ik} in the determinant A, which is likewise a collineation, the *inverse* of T. Thus if T carries x into x', T^{-1} carries x' into x.

It follows from (1) and (2) that *a collineation defines a* (1, 1) *correspondence between the points of the whole plane.* But being linear in both sets of variables, either T or T^{-1} transforms a curve into another of the same order. In particular collinear points go into collinear points. In other words *a collineation also sets up a* (1, 1) *correspondence between the lines of the plane.*[1] Furthermore the double ratio of a range of four points or a pencil of four lines is invariant under a collineation for the argument of §62 applies without change when the equations are interpreted as a collineation. In short, *equations* (1) *are the analytic representation of a projection of the plane upon itself.*

147. Some properties of a collineation.—The general ternary collineation contains eight essential constants. For any one of the nine homogeneous constants (a_{ik}) in its equations can be divided out. If follows that

1°. *A collineation is uniquely determined when four pairs of corresponding points, no three of which in either set are collinear, are specified.*[2]

For a point possesses two essential constants and to

[1] Hence the name collineation which is used in any dimension. *Cf.* the use of *linear*.

[2] This is implied by the fact that a projective coördinate system is completely determined when the triangle of reference and the unit point have been chosen. As soon as the vertices of the new triangle and the unit point in the new system have been expressed in terms of the old coördinates, the renaming of all points in the plane is determined. And abstractly a collineation and a change of coördinates are identical since both are expressed by the same linear transformation.

transform any point into another uses up two of the disposable constants in the collineation. If then four points x are independent,—*i. e.*, if no three are on a line,—we have just enough constants to send them into any four independent points x'. And since all the conditions involved are linear the determination is unique.[1]

In the language of projection theorem 1° may be stated in the useful form: *Four independent points in the plane can be projected into any other four independent points.*

We examine the collineation now for *fixed points, i. e.*, points which are left unaltered in position by the transformation. Such points are self-corresponding and must satisfy equations (1) §146 when $x' = x$, whence

$$\begin{aligned} -\rho x_1 + a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= 0 \\ -\rho x_2 + a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= 0 \\ -\rho x_3 + a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= 0. \end{aligned} \tag{1}$$

If these equations are to be consistent we must have, eliminating x

$$\begin{vmatrix} a_{11} - \rho & a_{12} & a_{13} \\ a_{21} & a_{22} - \rho & a_{23} \\ a_{31} & a_{32} & a_{33} - \rho \end{vmatrix} = 0 \tag{2}$$

a cubic in ρ, called the *characteristic equation* of the collineation. Each value of ρ in this equation leads to a fixed point, for equations (1) corresponding to any value of ρ are consistent and must have a solution. Hence

2°. *Every collineation has at least one fixed point.*

The general collineation in fact has three distinct fixed points which form a triangle. Let us find the form of the collineation referred to the triangle of fixed points which is a natural triangle of reference. We determine the coeffi-

[1] Three points of a line have but 5 constants, two to fix the line and one for each point. Hence to send 4 points, 3 of them collinear, into 4 others (3 collinear) would specify but 7 of the 8 available constants.

cients of the collineation (1) §146 by asking that (1, 0, 0), (0, 1, 0), (0, 0, 1) correspond respectively to (k_1, 0, 0), (0, k_2, 0), (0, 0, k_3).[1] Substituting (1, 0, 0) for (x_1, x_2, x_3) and (k_1, 0, 0) for (x_1', x_2', x_3') in the equations of the collineation, we find as the conditions that the first vertex correspond to itself

$$a_{11} = \rho k_1, \qquad a_{21} = 0, \qquad a_{31} = 0.$$

Likewise the conditions that the other vertices remain fixed are

$$\begin{array}{lll} a_{12} = 0, & a_{22} = \rho k_2, & a_{32} = 0 \\ a_{13} = 0, & a_{23} = 0, & a_{33} = \rho k_3. \end{array}$$

Thus $a_{11} : a_{22} : a_{33} = k_1 : k_2 : k_3$ and we may say

3°. *By taking the fixed points for vertices of the reference triangle the general collineation can be written in the canonical form*

$$\rho x_1' = k_1 x_1, \qquad \rho x_2' = k_2 x_2, \qquad \rho x_3' = k_3 x_3. \tag{3}$$

The k's are simply the roots of the characteristic equation

$$(\rho - k_1)(\rho - k_2)(\rho - k_3) = 0. \tag{4}$$

If (3) have a fourth fixed point say (a_1, a_2, a_3) not lying on a side of the reference triangle we must have $k_1 : k_2 : k_3 = 1$ and the collineation reduces to

$$\rho x_1' = x_1, \qquad \rho x_2' = x_2, \qquad \rho x_3' = x_3 \tag{5}$$

which is called the *identical collineation*, or *identity*. Obviously the identical collineation transforms every point into itself, hence

Cor. *If a collineation leave invariant each of four points no three of which are collinear, it leaves invariant every point of the plane.*

[1] The points are the same as the others since only the ratios are of consequence.

148. The planar homology.—If two (and only two) of the k's become equal in (3) of the last section, the collineation is called a *homology*. If $k_1 = k_2 \neq k_3$ we have at once the homology in canonical form

$$\rho x_1' = k_1 x_1, \qquad \rho x_2' = k_1 x_2, \qquad \rho x_3' = k_3 x_3.$$

Or dividing out the k_1 which may be done without loss of generality, we may write the homology

$$\rho x_1' = x_1, \qquad \rho x_2' = x_2, \qquad \rho x_3' = k_3 x_3. \tag{1}$$

The characteristic equation is

$$(\rho - 1)^2(\rho - k_3) = 0. \tag{2}$$

Hence if a collineation become a homology the characteristic equation has equal roots. The converse is however not true as we shall see below. In other words *a necessary but not sufficient condition for a homology is that the characteristic equation have equal roots.*

First we shall seek the fixed point corresponding to the double root $\rho = 1$. Setting $\rho = 1$ and $x' = x$ in (1) we get

$$x_1 - x_1 = 0, \quad x_2 - x_2 = 0, \quad (1 - k_3)x_3 = 0. \tag{3}$$

The first two equations in (3) vanish identically so that the three are satisfied by any point whose x_3-coördinate is zero. Consequently *every point on the line $x_3 = 0$ is a fixed point.* This line is called the *axis* of the homology.

Again replacing ρ by k_3, the simple root of the characteristic equation, we have as equations to determine the corresponding fixed point

$$(k_3 - 1)x_1 = 0, \quad (k_3 - 1)x_2 = 0, \quad k_3(x_3 - x_3) = 0. \tag{4}$$

Now the third equation vanishes identically so that the point of intersection (0, 0, 1) of the lines $x_1 = 0$, $x_2 = 0$ is a fixed point. Moreover any line on this point, called the *center* of the homology, is fixed. For any such line contains two fixed points, namely the center and the point in which the line cuts the axis. Thus

A homology has a line of fixed points, the axis, and a point of fixed lines, the center.

Cor. *A homology can have no other fixed elements unless it reduce to the identity.*

Proof. It cannot have another fixed point in virtue of the corollary, §147. Suppose now that it had another fixed line. Then the points of this line would be shifted among themselves in a (1, 1) way. That is the homology would effect on the points of the line a *binary* collineation which must have at least one fixed point (§81). But we have just proved this to be impossible.

It appears now that *a reflexion* (§144) *is a special case of a homology,* obtained by setting $k_3 = -1$ in (1).

EXERCISES

1. Find the fixed points and fixed lines of each of the collineations

$$(a)\ \begin{aligned} x_1' &= x_3, \\ x_2' &= x_1, \\ x_3' &= x_2, \end{aligned} \qquad (b)\ \begin{aligned} x_1' &= \omega^2 x_3, \\ x_2' &= x_1, \\ x_3' &= \omega x_2, \end{aligned} \qquad (c)\ \begin{aligned} x_1' &= \omega x_2 \\ x_2' &= \omega^2 x_3 \\ x_3' &= x_1 \end{aligned} \qquad \omega^3 = 1.$$

2. Write the collineation that

(*a*) leaves each vertex of the triangle of reference fixed and transforms the unit point into (2, −1, 3).

(*b*) leaves the unit point fixed and permutes the vertices of the reference triangle cyclically

(*c*) interchanges (1, 0, 0) with (1, 1, 1) and (0, 1, 0) with (0, 0, 1)

(*d*) advances cyclically (from left to right) the points (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1).

Show that (*c*) is a reflexion and find the center and axis. Find the fixed points and lines of (*b*) and (*d*).

3. Given the four points (1, 1, 1), (1, −1, 1), (1, 1, −1), (−1, 1, 1). Find the collineation that

(*a*) transforms (1, 1, 1) into (1, 0, 0) and leaves each of the other points fixed

(*b*) leaves the unit point fixed and advances cyclically (from left to right) the remaining

(c) interchanges (1, 1, 1) with (1, −1, 1) and (1, 1, −1) with (−1, 1, 1)

(d) advances cyclically the four points.

Find the fixed points and fixed lines of (b). Show that (c) is a reflexion and find its center and axis.

4. Write the inverse of each collineation in Exs. 2, 3.

5. If the line $(ux) \equiv u_1x_1 + u_2x_2 + u_3x_3 = 0$ is transformed by the collineation of §146 into $(u'x') = 0$, show that the relations between u and u' are

$$\begin{aligned} \sigma u_1' &= A_{11}u_1 + A_{12}u_2 + A_{13}u_3 \\ \sigma u_2' &= A_{21}u_1 + A_{22}u_2 + A_{23}u_3 \quad A_{ik} \neq A_{ki} \\ \sigma u_3' &= A_{31}u_1 + A_{32}u_2 + A_{33}u_3 \end{aligned}$$

where $\sigma = A/\rho$. We have thus the collineation expressed in line coördinates. Write the inverse of the collineation just obtained.

6. The collineation (1) §146 transforms a pencil of lines (or points) into a pencil projective with the original.

7. Extend the results of §§146–8, including the canonical equations, to space.

149. Classification and canonical equations of non-singular collineations.—We have already noticed two types of collineations,—(I) the general collineation and (II) the homology. The further study of possible types is facilitated by the following observations:

1°. *If a side $x_i = 0$ of the triangle of reference is a fixed line, then one equation of the transformation may be written* $\rho x_i' = k_ix_i$. For x_i' must reduce to zero when $x_i = 0$.

2°. *The ternary collineation effects a binary collineation on the points of any fixed line.* For the points of the line are shifted among themselves in a one-to-one way. Thus the binary transformation on the line $x_3 = 0$ (§146, (1)) is when this line is fixed

$$\rho x_1' = a_{11}x_1 + a_{12}x_2, \quad \rho x_2' = a_{21}x_1 + a_{22}x_2. \tag{1}$$

This binary collineation may be involutory or parabolic, or it may be the identity.

A convenient basis for the classification of collineations is the configuration formed by the fixed elements. We take throughout as the standard equations (1), §146.

Type III. Suppose two of the fixed points coincide at the vertex (0, 0, 1) of the reference triangle, approaching coincidence along the axis $x_1 = 0$ while the other fixed point remains distinct at the vertex (1, 0, 0). $x_1 = 0$ and $x_2 = 0$ must then be fixed lines which requires $a_{12} = a_{13} = a_{21} = a_{23} = 0$. Since (1, 0, 0) is a fixed point we must have in addition (§147) $a_{31} = 0$. If now k_1 is the root of the characteristic equation corresponding to the simple fixed point (1, 0, 0) the collineation becomes

$$\rho x_1' = k_1 x_1, \quad \rho x_2' = a_{22} x_2, \quad \rho x_3' = a_{32} x_2 + a_{33} x_3.$$

But the characteristic equation must have a double root, say k_3, corresponding to the two coincident fixed points, *i. e.*, we must have

$$(k_1 - \rho)(a_{22} - \rho)(a_{33} - \rho) \equiv (k_1 - \rho)(k_3 - \rho)^2, \text{ or } a_{22} = a_{33} = k_3.$$

Hence the canonical equations of type III are

$$\begin{aligned} \rho x_1' &= k_1 x_1 \\ \rho x_2' &= k_3 x_2 \qquad\qquad k_1 \neq k_3, \quad a_{32} \neq 0 \\ \rho x_3' &= a_{32} x_2 + k_3 x_3. \end{aligned} \tag{2}$$

Type IV. Let all three fixed points coincide at (0, 0, 1), k_3 being the corresponding (triple) root of the characteristic equation. Then (§147) $a_{13} = a_{23} = 0$ and the characteristic determinant is

$$\begin{vmatrix} a_{11} - \rho & a_{12} & 0 \\ a_{21} & a_{22} - \rho & 0 \\ a_{31} & a_{32} & a_{33} - \rho \end{vmatrix}.$$

But this must reduce to $(k_3 - \rho)^3$, *i. e.*,

$$(a_{33} - \rho)\{(a_{11} - \rho)(a_{22} - \rho) - a_{12}a_{21}\} \equiv (k_3 - \rho)^3$$

whence $a_{11} = a_{22} = a_{33} = k_3$ and either $a_{12} = 0$ or $a_{21} = 0$.

Hence type IV can be represented in the canonical form

$$\begin{aligned} \rho x_1' &= k_3 x_1 \\ \rho x_2' &= a_{21}x_1 + k_3 x_2 \qquad\qquad a_{21}, a_{32} \neq 0 \\ \rho x_3' &= a_{31}x_1 + a_{32}x_2 + k_3 x_3 \end{aligned} \tag{3}$$

where k_3 may be divided out if desirable.

$x_1 = 0$ is obviously a fixed line of the collineation (by 1°). It is indeed the only fixed line, for if there were another the two would intersect in a second fixed point whereas the collineation has but one.

Type V. Suppose now that equations (3) of type IV are restricted so that every point on the line $x_1 = 0$ is fixed, *i. e.*, the collineation on x_1 is the identity. Then we must have (by 2°) $a_{32} = 0$ and the collineation is

$$\rho x_1' = k_3 x_1, \quad \rho x_2' = a_{21}x_1 + k_3 x_2, \quad \rho x_3' = a_{31}x_1 + k_3 x_3.$$

Then every line of the pencil $\lambda x_1 + a_{31}x_2 - a_{21}x_3 = 0$ is a fixed line as may be readily verified. The center $(0, a_{21}, a_{31})$ of this pencil may be taken as vertex $(0, 0, 1)$ when $a_{21} = 0$ and the collineation becomes

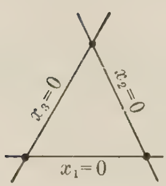
$$\begin{aligned} \rho x_1' &= k_3 x_1 \\ \rho x_2' &= k_3 x_2 \\ \rho x_3' &= a_{31}x_1 + k_3 x_3 \end{aligned} \tag{4}$$

which is called an *elation.*

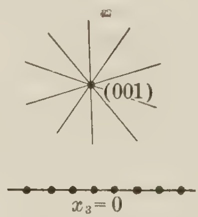
Thus the elation like the homology has an axis of fixed points and a center of fixed lines but unlike the homology the center and axis of the elation are incident.

To summarize we have (exclusive of the identity) five types of non-singular collineations characterized as follows:

Type	Fixed Elements	Characteristic Equation Has
I	Three fixed points and three fixed lines forming a triangle.	three distinct roots.
II	Line of fixed points and point of fixed lines.	double root.
III	Two fixed points, two fixed lines.	double root.
IV	One fixed point and one fixed line which are incident.	triple root.
V	Line of fixed points and point of fixed lines, center and axis incident.	triple root.



I



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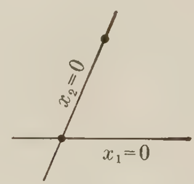

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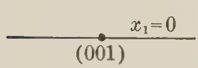
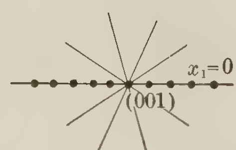


IV



V

The configurations of the fixed elements are indicated in the accompanying diagram.

If the determinant A (§146) is zero, the collineation is no longer reversible. It is then said to be *singular*. For a classification of singular collineations the student may consult Bôcher, *Introduction to Higher Algebra*, p. 294.

EXERCISES

1. Show that the various types of collineations can have no fixed lines other than those enumerated. (Two fixed lines must meet in a fixed point.) Hence show that the configuration of fixed elements is self-dual.

2. Write the equations of the binary collineation on each fixed side of the reference triangle in types I–V, also for the reflexion. What is the type of each binary collineation.

3. If a collineation effects the identical collineation on two sides of the reference triangle it is the identical collineation.

4. An elation is a special case of a reflexion.

5. Write the most general collineation (in homogeneous Cartesian coördinates) that leaves the line at infinity invariant.

6. Write the equations of a collineation whose fixed points are the origin and the circular points.

7. Show that $x' = x + hz$, $y' = y + kz$, $z' = z$ represent an elation and find the center and axis.

8. Classify as far as possible the singular collineations. The classification will depend on the way the matrix of the collineation becomes zero. Thus A may be zero without there being any relation between any two coefficients, the coefficients in two rows or two columns may be proportional, the coefficients in a row or column may all be zero, etc. Characterize the types geometrically, as, *e. g.*, one point is fixed and all other points of the plane go into points on a line. (Bôcher, *l. c.*, notes 8 types including the extreme case when every element of the matrix vanishes.)

9. Write in line coördinates the canonical equations of the collineations I–V. Show that each effects a binary collineation on the lines of the pencil determined by any fixed point. Classify these binary collineations.

150. Product of collineations.—We shall now consider the effect of a composition of collineations, introducing as a preliminary the convenient notion of the square matrix. A collineation

$$S:\quad \begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{aligned}$$

is sometimes denoted symbolically by the array of coefficients on the right

$$A \equiv \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

which is called the *matrix* of the collineation. Such an array is a mere symbol which has no numerical or functional value and is not to be confused with the determinant of the collineation. Thus a matrix cannot be expanded like a determinant, neither can its rows and columns be interchanged for the symbol would no longer represent the same collineation.

Let S be a collineation (with matrix A) which carries the point x into the point x' while

$$T: \quad \begin{aligned} x_1'' &= b_{11}x_1' + b_{12}x_2' + b_{13}x_3' \\ x_2'' &= b_{21}x_1' + b_{22}x_2' + b_{23}x_3' \\ x_3'' &= b_{31}x_1' + b_{32}x_2' + b_{33}x_3' \end{aligned} \qquad (2)$$

is a collineation (with matrix B) which carries the point x' into x''. Then the operation that carries x into x'' we shall call the *resultant* or *product* TS of S and T since the operation is obviously effected by an application of S followed by an application of T.[1] Substituting in (2) the values of x' from (1) we obtain as the product

$$\begin{aligned} x_1'' = &\quad (a_{11}b_{11} + a_{21}b_{12} + a_{31}b_{13})x_1 \\ &+ (a_{12}b_{11} + a_{22}b_{12} + a_{32}b_{13})x_2 \\ &+ (a_{13}b_{11} + a_{23}b_{12} + a_{33}b_{13})x_3, \\ x_2'' = &\quad (a_{11}b_{21} + a_{21}b_{22} + a_{31}b_{23})x_1 \end{aligned}$$

[1] The student will note that the multiplication as here defined is from right to left, *i. e.*, the collineation operating first is placed on the right. This is the practice with polar and other functional operators and is easily remembered since we understand that a collineation transforms the old variables on the right into the new variables on the left. Either order of course could be chosen but once chosen it must be followed since multiplication is not commutative in general.

$$
\begin{aligned}
TS: \qquad & \quad + (a_{12}b_{21} + a_{22}b_{22} + a_{32}b_{23})x_2 \qquad (3)\\
& \quad + (a_{13}b_{21} + a_{23}b_{22} + a_{33}b_{23})x_3,\\
x_3'' = & \quad\ (a_{11}b_{31} + a_{21}b_{32} + a_{31}b_{33})x_1\\
& \quad + (a_{12}b_{31} + a_{22}b_{32} + a_{32}b_{33})x_2\\
& \quad + (a_{13}b_{31} + a_{23}b_{32} + a_{33}b_{33})x_3
\end{aligned}
$$

which is again manifestly a collineation.

The matrix of the product TS defines the product BA of the matrices of T and S. Thus *the element in the ith row and kth column of the product BA is formed by multiplying each element of the ith row of B by the corresponding element of the kth column of A and adding the results.*[1] Hence

1°. *The product TS of two collineations S and T is a collineation whose matrix is the product of the matrices of T and S.*

2°. *Multiplication of collineations is not in general commutative.*

For the matrix of ST is AB which in accordance with the definition above is different from BA in the general case. Geometrically this means that if the combined effect of S followed by T is to carry a point x into a point y then the effect of T followed by S is to carry x into some point other than y.

3°. *Multiplication of collineations is associative.*

If R carries x into y, S carries y into z and T carries z into x' then as in §85 we may denote the effect of the product TSR by ${}_{x'}T_zS_yR_x$. Likewise the effect of $T(SR)$ is ${}_{x'}T_z(SR)_x$ while the effect of $(TS)R$ is ${}_{x'}(TS)_yR_x$. In either case the net result is to carry the point x into the point x' which proves the theorem. Symbolically, $TSR = T(SR) = (TS)R$.

[1] If the elements of a row are numbered from left to right and those of a column downward then corresponding elements are those with the same serial number. *Cf.* the rule for multiplying determinants §105. While there are four methods for forming the product of two determinants, the rule for multiplying matrices is unique.

Corollaries.—The identical collineation is denoted by 1. Since the identity leaves every point unaltered

$$(a) \qquad S1 = 1S = S.$$

If S^{-1} is the inverse of a collineation S we have from the definition (§146) $S^{-1}S = 1$. That is the effect of applying S^{-1} to the points of the plane already transformed by a collineation S is to restore each point to its original position.

$$(b) \qquad S^{-1}S = 1 = SS^{-1}$$

i. e., the inverse relation is reciprocal. For if S carries x into x' then S^{-1} carries x' into x and it is evidently immaterial which transformation is applied first in the product.

$$(c) \qquad \text{If} \qquad XS = XT, \qquad \text{then} \qquad S = T.$$

For multiplying both sides by X^{-1} we have $X^{-1}XS = X^{-1}XT$, or $S = T$. Likewise if $SX = TX$, $S = T$. (Multiply $X^{-1} = X^{-1}$ by $SX = TX$ and apply (*b*).)

If a collineation S is applied repeatedly, say n times the resulting collineation is a *power* of S, denoted by S^n. The inverse of S^n is S^{-n}. If $S^n = 1$, S is said to be of *period n.* From the associative law (3°) and (*b*) it follows that

(*d*) $S^iS^k = S^kS^i = S^{i+k}$, i, k any positive or negative integer.

The product TST^{-1} is called the *transform* of S by T.

(*e*) *If S carries x into x' and T carries x into y (and x' into y') then the effect of the transform is ${}_{y'}T_{x'}S_xT_y{}^{-1}$, i. e., the transform carries y into y'.*

This statement holds when S and T are both collineations, both transformations of coördinates, one a collineation and the other a transformation of coördinates. If S represents a collineation and T a transformation of coördinates then the transform causes the same shifting of the points of the plane as S but expresses it in a new coördinate system.

151. Invariants of a collineation.—The roots k_i of the characteristic equation are called *multipliers* of the collineation. If (x_1, x_2, x_3) is a fixed point of a collineation S then S carries the point into (kx_1, kx_2, kx_3), where k is one root of the characteristic equation. Or we may say briefly that S carries x into kx. If then T is a collineation that changes x into y the effect on x when S is transformed by T is (§150) ${}_{ky}T_{kx}S_xT_y{}^{-1}$, *i. e.*, the transform carries y into ky. Hence T transforms the fixed points of S into fixed points of the transform. Moreover the multipliers are unchanged when the collineation is transformed. In other words

The multipliers are invariants of the collineation.

They are however irrational invariants in general since they are roots of a cubic. Evidently any symmetric function of the k's will also be an invariant. Thus the coefficients in the characteristic equation are rational, integral invariants, in fact they are a complete system of such invariants. Consequently we may write the characteristic determinant

$$\begin{vmatrix} a_{11}-\rho & a_{12} & a_{13} \\ a_{21} & a_{22}-\rho & a_{23} \\ a_{31} & a_{32} & a_{33}-\rho \end{vmatrix} \equiv -(\rho^3 - I_1\rho^2 + I_2\rho - I_3)$$

where I_i is an invariant of degree i in the coefficients.

Since the transform of S by T may either express the collineation in a new coördinate system or it may represent the collineation under a projection of the plane, the vanishing of an invariant implies a property of the collineation which is at once independent of the coördinate system, and unaltered by projection.

EXERCISES

1. Given the collineations

$$S:\ x_i' = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3$$
$$T:\ x_i' = b_{i1}x_1 + b_{i2}x_2 + b_{i3}x_3 \qquad i = 1, 2, 3.$$

The product TS is found by substituting for x_i in T the values of x_i' from S. Likewise the product ST is found by substituting for x_i in S the values of x_i' from T. Extend the rule to three collineations S, T, U (with matrix C). State a rule for finding the square, the cube of a collineation by substitution.

2. Write the square and the cube of each collineation in Ex. 1, §148. Also write the product of each pair of collineations in both orders and the product of all three in all six orders.

3. Show that the square of the collineation

$$\begin{aligned} x_1' &= x_1 + x_2 + x_3 \\ x_2' &= x_1 + \omega x_2 + \omega^2 x_3 \\ x_3' &= x_1 + \omega^2 x_2 + \omega x_3 \end{aligned}$$

is a reflexion and hence (or directly) the collineation is of period 4.

4. Write the cube of the collineation of type I in canonical form ((3), §147). What are the conditions that the collineation be of period 3, period n?

5. Show by multiplication of the matrices or by substitution that the collineations (b) and (d) of Ex. 2, §148 are of period 3 and 4 respectively.

6. Show by multiplying the matrices that the product of a collineation T (§146) and its inverse T^{-1} is the identity.

7. If S and T represent any two collineations and if $TST^{-1} = S^{-1}$, show that $TS = S^{-1}T$. Also $TS^2 = TSS = S^{-1}TS = S^{-1}S^{-1}T = S^{-2}T$ and generally $TS^k = S^{-k}T$.

8. Expand the characteristic determinant of the general collineation and find the values of the invariants (§151).

Ans. $I_1 = a_{11} + a_{22} + a_{33}$, $I_2 = A_{11} + A_{22} + A_{33}$, $I_3 = A$.

What is the effect on the characteristic equation of the vanishing of the individual invariants?

9. Write the invariant condition that the characteristic equation have a double root, a triple root. (See §114 and §115, Ex. 20.)

10. Write the invariants for each of the collineations $I - V$, including the case of the reflexion. Obtain relations connecting the invariants for each type and thus characterize the various types. (Make use of Ex. 9 and the fact that invariant relations must be homogeneous in the coefficients.)

11. Find the invariants and thence expand the characteristic determinant of the collineation in line coördinates (§148, Ex. 5). *Ans.* $-\sigma^3 + I_2\sigma^2 - AI_1\sigma + A^2$. Hence show, since $\sigma = A/\rho$, that the determinant differs only by a factor $(-A^2)$ from the characteristic

determinant for the point form. Hence the configuration of fixed points and fixed lines of a collineation is self-dual.

12. Write out in full the transform TST^{-1} of S by T, Ex. 1.

13. Obtain a solution of Ex. 3(a) §148 as follows: Write the collineation when the fixed points are vertices of the reference triangle (Ex. 2(a) in the same set) and then pass to the original triangle of reference by means of the transform. Solve the other parts of the exercise in a similar way.

14. Find the invariants (coefficients in the characteristic equation) of the general collineation in space.

15. The equations

$$\begin{aligned} \sigma u_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ \sigma u_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \qquad a_{ik} \neq a_{ki} \\ \sigma u_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \qquad A \neq 0 \end{aligned} \tag{1}$$

define a *correlation* of matrix A, *i. e.*, a (1, 1) correspondence between the points x and the lines u of the plane. We regard the correlation (1) as the operation that transforms the point x into the line u. Obtain the inverse (correlation) that carries u back into x by solving (1) for the x's.

16. Defining the product of two correlations or of a correlation and a collineation as in the case of two collineations, show that the product of two correlations is a collineation, while the product of a correlation and a collineation is a correlation. In either case the matrix of the product is the product of the matrices of the component transformations.

17. The correlation (1) also carries lines into points. If it transforms the line $(ux) \equiv u_1x_1 + u_2x_2 + u_3x_3 = 0$ into the point $(x'u') = 0$, the relations between the coördinates of the line u and the point x' into which it is transformed are

$$\begin{aligned} \rho x_1' &= A_{11}u_1 + A_{12}u_2 + A_{13}u_3 \\ \rho x_2' &= A_{21}u_1 + A_{22}u_2 + A_{23}u_3 \qquad . A_{ik} \neq A_{ki} \\ \rho x_3' &= A_{31}u_1 + A_{32}u_2 + A_{33}u_3. \end{aligned}$$

18. A polarity (§145) is a special case of the general correlation. The necessary and sufficient condition that the general correlation be a polarity is that its matrix be symmetrical, *i. e.*, $a_{ik} = a_{ki}$.

19. (1) Ex. 15 gives the coördinates of the line into which the correlation transforms a point. Find the *equation* of this line and thus express the correlation as a general ternary bilinear form.

20 Given the collineations
$S: x_1' = x_1,\ x_2' = \omega x_2,\ x_3' = \omega^2 x_3$, and $T: x_1' = x_2,\ x_2' = x_1,\ x_3' = x_3$ and the polarity Π: $u_1 = x_1,\ u_2 = x_2,\ u_3 = 10x_3$. Find the products ΠS, ΠS^2, ΠT, ΠST, ΠS^2T. Select the polarities among the products and find the base conic of each. What are the periods of the remaining correlations?

21. The locus of a point whose corresponding line, under a correlation, is on the point is a conic C_p and the locus of a line whose corresponding point is on the line is a conic C_l. Find the equations of these conics and show hence that each is the polar reciprocal of the other with respect to $x_1^2 + x_2^2 + x_3^2 = 0$ (*i. e.*, the curves are dual).

22. If a correlation is involutory, *i. e.*, if the point x goes into the line u and the line u goes back into x, it is a polarity.

23. The conics C_p and C_l (Ex. 21) for a polarity coincide with the base conic of the polarity.

24. The general correlation has two fixed conics; a polarity one.

152. Group of collineations.—Consider the set of four collineations consisting of the identity and three reflexions

	R_1	R_2	R_3	1
$x_1' =$	$-x_1$	x_1	x_1	x_1
$x_2' =$	x_2	$-x_2$	x_2	x_2
$x_3' =$	x_3	x_3	$-x_3$	x_3.

An examination of these collineations will reveal the following properties

1°. *The product of any two of the collineations is a collineation of the set.*[1]

2°. *The inverse of each collineation occurs in the set.*[2]

3°. *The identical collineation is one of the set.*

A set of collineations satisfying these three requirements is said to form a *group* of which the constituent collineations are the *elements*.[3] The *order* of a group is

[1] Thus $R_iR_j = R_k$, $R_i^2 = 1$, $1R_i = R_i$.

[2] Each collineation is in fact its own inverse.

[3] In the theory of abstract groups it is further stipulated:

4°. *The associative law holds for the product of any three elements.*

This is superfluous for present purposes since we have proved that multiplication of collineations is of necessity associative. Further in the

the number of collineations which it contains. A group may be either *finite* or *infinite* according as the order is finite or infinite. The group under consideration is thus a finite group of order four and may be denoted by G_4. It is known as the ternary dihedral G_4.

153. Some properties of the dihedral G_4.—The effect of our G_4 is to carry an arbitrary point of the plane into (at most) four positions including the original position. For each collineation either leaves the point fixed or transforms it into a second point. Hence the point may occupy four distinct positions. It cannot occupy more since by 1° two of the collineations successively applied produce the same effect as some one of the collineations. Thus the points of the plane under the group are arranged in sets of four, the four points in any set being *conjugate*. That is, a set of conjugate points consists of those points which can be transformed into each other by the operations of the group.

Some points however may assume fewer than four positions under the group. Thus the center and axis of each reflexion, namely a vertex and opposite side of the reference triangle, are a point of fixed lines and a line of fixed points under that reflexion (§144). Consequently the vertices and sides of the triangle of reference are fixed or *invariant* under the whole group. Or

---

abstract theory some law of combination is hypothecated. The elements may have any meaning so long as they are capable of being combined according to some law. If we replace "collineation" by "element," 1°-4° are postulates for abstract groups. The result of combining one element of an abstract group with another is called a *product* of the two even though actually it may be in a concrete case addition, permutation, substitution, rotation, etc. We say, *e. g.*, that the positive and negative integers and zero "form a group with respect to addition." The integers are here the elements and addition is the law of combination.

Ex. The identical element is 0. (The identity must satisfy Cor. (*a*) §150.)

Two groups with the same abstract structure are said to be isomorphic.

The group has an invariant triangle whose vertices are the centers and whose sides are the axes of reflexion.

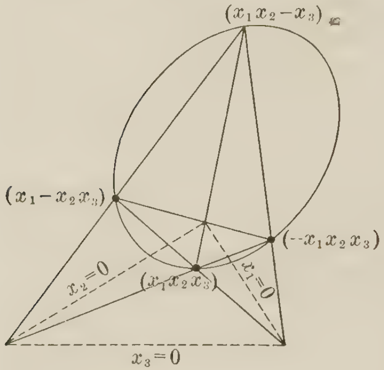
To find the invariant conics we observe that the effect of R_i is to change the sign of those terms containing x_i in the first degree while it leaves other terms unaltered. Thus every conic in the system

$$ax_1^2 + bx_2^2 + cx_3^2 = 0$$

is invariant under the group. Or

Each conic of the net apolar to the invariant triangle is an invariant of the group.

Consider now a particular invariant conic. In general the points of the conic lie in conjugate sets of four. The only exceptions are those points which are invariant under some collineation of the group. Thus a point on an axis is fixed under one reflexion and can take up in consequence only two positions. In other words



The conic contains three special conjugate sets, namely the three pairs of points cut out by the axes.[1]

A line from a center of reflexion to one of a set of four conjugate points cuts out a second. Hence the points lie on three pairs of lines one pair meeting at each center. Therefore

The four conjugate points of a set lie at the vertices of a quadrangle whose diagonal triangle is the invariant triangle.

[1] These special sets may be considered as sets of four points which coincide in pairs.

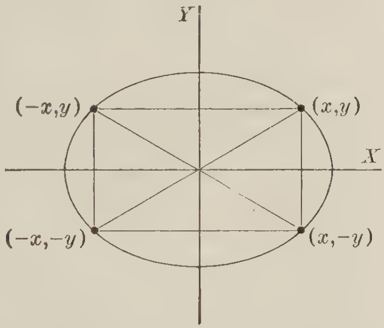
Since each point of the conic determines a set of four conjugate points and there are ∞^1 points on the curve we see again that a conic has a single infinity of inscribed quadrangles with a common diagonal triangle.[1]

The G_4 *in metrical form* can be found by taking the Cartesian axes and the line at infinity for the invariant triangle. The transformations then are

	R_1	R_2	R_3	1
$x' =$	$-x$	x	$-x$	x
$y' =$	y	$-y$	$-y$	y

where x, y are rectangular coördinates.

The properties of the group obtained above are now obvious. Four conjugate points have the coördinates $(\pm x, \pm y)$, *i. e.*, the points are vertices of a rectangle whose center is the origin and whose sides are parallel to the axes.



The net of invariant conics consists of the central conics

$$ax^2 + by^2 + c = 0$$

each of which is symmetrical with respect to the axes as well as the origin. Indeed any invariant curve of the group in this form has the origin for center (point of symmetry) and the coördinate axes for lines of symmetry.

154. Subgroups. Generators. Cyclic groups.—Sometimes a part of the elements of a group constitute a group in themselves. We then have a group within a group which is called a *subgroup* of the original. For example the iden-

[1] *Cf.* the theory of the binary quartic §§120, 134.

tity and any one of the reflexions of G_4, §152, form a subgroup of the G_4. It is readily proved[1] that

The order of every subgroup is a factor of the order of the group.

It usually happens that a very few of the elements of a group determine by their products all other elements. Such a set of elements generate the group and are called *generators*. Thus in the dihedral G_4 of the last section $R_1R_2 = R_3$ and $R_1^2 = 1$ so that R_1 and R_2 (or any two of the R's) are generators.

A group that can be generated by a single element say S is called a *cyclic* group. A cyclic group is necessarily finite (since some power of S must be the identity) and consists wholly of powers of the generating element thus

$$S, S^2, S^3, \quad . \quad . \quad . \quad S^n = 1. \tag{1}$$

Conversely any set of elements like (1) form a cyclic group since they satisfy the definitions of a group (§152). Moreover the product of any two elements is commutative (Cor. *d*, §150) and the group is called *commutative* or *Abelian*.

Any element of a finite group G_n generates a cyclic subgroup of G_n.

For if S is an element of G every power of S is an element of the group (definition 1°). Consequently for some finite integer k we must have $S^k = 1$ or otherwise the group would be of infinite order contrary to hypothesis. Finally the inverse of each power S^i of S is a power of S, *viz.*, S^{k-i}, which completes the proof of the theorem. Two trivial cases occur, when $k = 0, n$. In the first case the subgroup is the identity, in the other it is G_n itself.

[1] See for example, Miller, Blichfeldt, Dickson, *Finite Groups*, p. 23.

EXERCISES

1. The following aggregates of elements constitute groups:

(a) all collineations in the binary domain (3-parameter)
(b) all collineations in the ternary domain (8-parameter)
(c) all collineations of determinant unity in both the binary and ternary domain
(d) all collineations and correlations in the plane
(e) all rotations of the plane about a fixed point
(f) all rotations of the plane about a fixed point of period n
(g) all the permutations of n letters
(h) all cyclic permutations of n letters
(i) the six double ratios of four points in the binary domain (with respect to substitution).

Select (1) the finite groups in the list and state the order of each and (2) the cyclic groups.

2. There is no group consisting entirely of correlations.

3. The only finite groups without subgroups (excluding the two trivial cases, *viz.*, the identity and the whole group) are cyclic groups of order n, n a prime number.

4. All subgroups of a cyclic group are themselves cyclic.

5. The ternary dihedral collineation group G_4 can be written

$$\begin{array}{lcccc} & R_1 & R_2 & R_3 & 1 \\ x_1' = & x_1 & x_3 & x_3 & x_1 \\ x_2' = & -x_2 & x_2 & -x_2 & x_2 \\ x_3' = & x_3 & x_1 & x_1 & x_3. \end{array}$$

(Show that $R_1^2 = R_2^2 = R_3^2 = 1$, $R_iR_j = R_k$.) Find the invariant triangle and the net of invariant conics.

6. Show that the two collineations S and T below

$$S: \begin{aligned} x_1' &= x_1 \\ x_2' &= \omega x_2 \\ x_3' &= \omega^2 x_3 \end{aligned} \qquad T: \begin{aligned} x_1' &= x_3 \\ x_2' &= x_1 \\ x_3' &= x_2 \end{aligned} \qquad U: \begin{aligned} x_1' &= x_1 \\ x_2' &= x_3 \\ x_3' &= x_2 \end{aligned}$$

generate a G_9. Find the elements of the G_9 and show that they can be arranged to form four cyclic subgroups of order 3. S, T and U generate a G_{18} containing, besides the identity, 9 elements of period 2 and 8 elements of period 3.

U generates with either S or T a G_6. The entire G_{18} leaves invariant the cubic curve

$$x_1^3 + x_2^3 + x_3^3 + 6ax_1x_2x_3 = 0.$$

The group contains 9 cyclic subgroups of order 2. Find the centers and axes of the reflexions and show that the centers lie on the curve. The group also leaves unaltered each of four triangles, *viz.*, the fixed triangles of the four cyclic G_3's. (Three of these are the triangles of Ex. 1, §148, the fourth is the reference triangle.) *Cf.* Ex. 10, §61.

7. Let $(abc\ .\ .\ .\ r)$ denote a cyclic permutation or *cycle* of the letters, such that a goes into b, b into c, . . . r into a. Likewise let $(abc)(de)$ denote a cyclic permutation of a, b, c and the interchange of d and e. $(abc)(d)$ indicates a cyclic advance of a, b, c while d remains fixed. Show that the two permutations (abc) and (ab) generate the G_6 of permutations of the letters a, b, c. This group contains the identity, two elements of period 3 and three elements of period 2.

8. The permutations of four things a, b, c, d constitute a G_{24}. The four permutations $abcd = 1$, $bcda = P$, $cdab = P^2$, $dabc = P^3$ evidently form a cyclic subgroup G_4. Now P^2 is obtained from 1 by advancing the letters cyclically *two* spaces, or by interchanging a with c and b with d. Similarly P^3 is obtained by advancing the letters in 1 three spaces. Thus the four permutations can be written in cycle form: $(a)(b)(c)(d) = 1$, $(abcd) = P$, $(ac)(bd) = P^2$, $(adcb) = P^3$. P and P^3 are of period 4 while P^2 is of period 2. Show that the G_{24} contains besides the identity

6 elements of period 4 like $(abcd)$
8 elements of period 3 like $(abc)(d)$
3 elements of period 2 like $(ab)(cd)$
6 elements of period 2 like $(ab)(c)(d)$.

The letters which are not permuted may be omitted from the cycles.

9. The rotations of a cube (a) of period 4 about a line joining the midpoints of two opposite faces, (b) of period 3 about a diagonal, (c) of period 2 about a line joining the midpoints of opposite edges constitute a G_{24} isomorphic with the group of permutations in Ex. 8. (The rotations permute in all possible ways the four diagonals.) This group is the same as the rotations of a regular octahedron into itself since the midpoints of the faces of a cube are vertices of a regular octahedron. The group is known as the *octahedral* group. If the cube (or its associated octahedron) is inscribed in a sphere then the group of rotations arranges the points of the sphere into general sets of 24 conjugate points. Enumerate the special sets of conjugate points (fewer than 24) and locate them on the sphere.

10. Enumerate the subgroups of the G_{24} in Ex. 8 (or 9). (The order of a subgroup is a factor of 24.) Defining *conjugate subgroups*

as a set of subgroups which are transformed into one another by the elements of the group we have:

Subgroups	Typical generators
3 conjugate cyclic G_4's	$(abcd)$
4 conjugate cyclic G_3's	(abc)
3 conjugate cyclic G_2's	$(ab)(cd)$
6 conjugate cyclic G_2's	(ab)
3 conjugate G_8's	$(abcd)$, $(ab)(cd)$
4 conjugate G_6's	(abc), (ab)
3 conjugate G_4's	(ab), (cd)
1 self-conjugate (*invariant*) G_4	$(ab)(cd)$, $(ac)(bd)$
1 invariant G_{12}	(abc), $(ab)(cd)$.

Ascertain the number of elements of period 2, 3, 4 in the various subgroups. Write all the permutations both in the ordinary and cycle form composing a subgroup of each type.

11. The rotations of a regular tetrahedron into itself consitute a group, the *tetrahedral* group. Enumerate the elements of various periods and find the order of the group. Enumerate the special sets of conjugate points and locate them on the circumscribing sphere.

12. The rotations of a sphere that transform into itself a regular inscribed icosahedron constitute the *icosahedral* group G_{60}. Find the number of elements of the various periods. Enumerate the special sets of conjugate points and locate them on the sphere. The rotations of a regular dodecahedron into itself is effected by the icosahedral group since the centers of the faces of a regular icosahedron are vertices of a regular dodecahedron.

13. The tetrahedral group is a subgroup of the octahedral group. Find as many subgroups of the tetrahedral and icosahedral groups as you can. (For a systematic account of the regular body groups, see Klein, Ikosaeder.)

155. The nth roots of unity.—We shall collect in this section certain elementary properties of the roots of unity, a knowledge of which is essential in subsequent sections. Any value of x that satisfies the equation

$$x^n = 1, \qquad n \text{ a positive integer} \tag{1}$$

is called an nth root of unity. While if n is the smallest positive integer for which $\epsilon^n = 1$, ϵ is said to be a *primitive*

nth root of *index n* of unity. Thus -1 is a fourth root of 1 but not a primitive fourth root since $(-1)^2 = 1$.

If ϵ is a primitive nth root the powers

$$\epsilon^0 = 1, \epsilon, \epsilon^2, \ldots \epsilon^{n-1} \tag{2}$$

are all distinct and comprise the whole set of n nth roots.

That they are all nth roots follows from the relation

$$(\epsilon^r)^n = (\epsilon^n)^r = 1^r = 1.$$

To prove that they are all distinct suppose $\epsilon^r = \epsilon^s$ where r, s are two exponents in (2), *i. e.*, $r, s < n$, $r > s$. Then dividing both sides by ϵ^s we have $\epsilon^{r-s} = 1$. But this contradicts the hypothesis that ϵ is a primitive nth root. Q. E. D.

Algebraically the roots can be found in simple cases and in all cases when n is a power of 2 by factoring. For example $x^4 - 1 = (x^2 - 1)(x^2 + 1) = (x - 1)(x + 1)(x - i)(x + i)$, whence the roots are ± 1, $\pm i$, $(i = \sqrt{-1})$. Again $x^8 - 1 = (x^4 - 1)(x^4 + 1) = (x^4 - 1)(x^2 - i)(x^2 + i) = (x^4 - 1)(x - \sqrt{i})(x + \sqrt{i})(x - \sqrt{-i})(x + \sqrt{-i})$. Hence the 8th roots are the 4th roots and $\pm\sqrt{i}$, $\pm\sqrt{-i}$.

Trigonometrically the nth roots are given by Demoivre's theorem in the form

$$\epsilon_k = \cos\frac{2k\pi}{n} + i \sin\frac{2k\pi}{n}, k = 0, 1, \ldots n - 1. \tag{3}$$

Or since (Euler's theorem) $\cos\theta + i\sin\theta = e^{i\theta}$

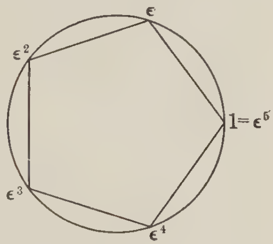
$$\epsilon_k = e^{\frac{2k\pi i}{n}}, \qquad k = 0, 1, \ldots n - 1$$

Roots of unity as operators.[1] The roots of 1 may be represented graphically in the usual manner on the complex plane. We shall now consider their interpretation as operators. For this purpose we think of t as the operator or *turn* that rotates the point 1 about the origin in the

[1] See Harkness and Morley, *Analytic Functions*, p. 14.

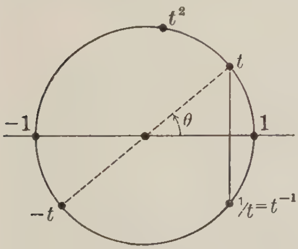
positive (counter-clockwise) sense through an angle θ. If as in group theory we define a successive application of t as the product of t by itself, then t^2 is a rotation through the angle 2θ. Or generally if t_1 is a rotation of 1 through θ_1 and t_2 a rotation through θ_2, t_1t_2 is a rotation through $\theta_1 + \theta_2$. The identical rotation which produces no effect we denote by $t^0 \equiv 1$.

If $t^2 = 1$, t will represent either the identity or a rotation through the angle π. But if $t^2 = 1$, $t = 1$ or -1, *i. e.*, -1 is a rotation through the angle π. Likewise if $t^3 = 1$, t may be (in addition to the identity) a rotation through either the angle $2\pi/3$ or $4\pi/3$. The corresponding values of t we denote by ω and ω^2, the imaginary cube roots of unity. Thus $1, \omega, \omega^2$ as turns carry 1 into the vertices of an equilateral triangle. Generally if n is the least positive integer for which $t^n = 1$, then one t is a rotation through the angle $2\pi/n$. If we denote this rotation by ϵ, then *the set of nth roots*



$$\epsilon, \epsilon^2, \epsilon^3, \quad . \quad . \quad . \quad \epsilon^n = 1$$

when interpreted as turns carry the point 1 successively into the vertices of a regular n-gon inscribed in the unit circle.[1]



The inverse of t, *i. e.*, t^{-1} is naturally a rotation of 1 through the angle θ in the negative direction. This may be defined as division of 1 by t which conforms to the algebra since $1/t = t^n/t = t^{n-1} = t^{-1}$. *Thus $-t$ is a reflexion of t in the origin*

[1] That the nth roots represent the vertices of such a polygon appears also from (3).

while $1/t$ is a reflexion of t in the axis of real numbers.

The following theorems are now geometrically evident:

The product (or quotient) of any two nth roots of unity is an nth root.

Any integral power, positive, negative or zero, of an nth root of 1 is an nth root.

The developments of this section can be summarized in the single statement:

Any primitive nth root of unity generates (with respect to multiplication) a cyclic group of order n whose elements are the whole set of nth roots. Geometrically, *the totality of rotations of a regular n-gon into itself constitute a cyclic group of order n.*

156. Absolute coördinates.—The developments of the preceding section are intimately associated with a second system of metrical coördinates. In this system we select for bilinear axes the pair of circular rays through the Cartesian origin. The coördinates of a point referred to these lines are called *circular* or *absolute* coördinates and are denoted by x and $\bar{x}$.

If X, Y are the usual rectangular coördinates of a point, the absolute coördinates of the point are given by the transformation

$$\begin{aligned} x &= X + iY \\ \bar{x} &= X - iY. \end{aligned} \qquad i = \sqrt{-1} \tag{1}$$

To express absolute coördinates in terms of rectangular we have but to solve equations (1), thus

$$X = \frac{x + \bar{x}}{2} \qquad Y = \frac{x - \bar{x}}{2i}. \tag{2}$$

Absolute coördinates may be made homogeneous if desirable by adding the line at infinity as a third axis of reference and writing

$$x = X + iY, \quad \bar{x} = X - iY, \quad z = Z \tag{3}$$

where z, Z is the line at infinity in either system. The non-homogeneous form is recovered at once by setting $z = 1$.

Absolute coördinates are specially adapted to discussion of metrical properties which is not surprising in view of the intimate relations that all metrical properties bear to the circular points. The equation of the circle $X^2 + Y^2 = r^2$ for example becomes in absolute coördinates $x\overline{x} = r^2$.

The process of representing these coördinates geometrically demands careful attention. Since the absolute coördinates are complex numbers they are plotted as in the complex plane. But in this plane a single complex number will locate a point and x and $\overline{x}$ if plotted separately would in general determine two points. To avoid this ambiguity we plot only the x. The $\overline{x}$ is however not superfluous as we shall see.

Imaginary points. Here as elsewhere we shall say that a point which can be represented in the plane of operations, *i. e.*, one that can actually be plotted is real, otherwise imaginary. But the usual algebraic criterion[1] for imaginary points will no longer suffice since the absolute coördinates of points that can be plotted are in general imaginary. One or two illustrations will make clear the distinction between real and imaginary points in the present system. Consider the line

$$x + \overline{x} = 4, \text{ or } X = 2. \tag{4}$$

It is evident geometrically, or it may be inferred directly from (1) that the only points which can be represented are those whose x-coördinates are of the form $2 + iY$. The corresponding $\overline{x}$-coördinates are then $2 - iY$. In other

[1] §20. In general if the reference framework is real an imaginary point is one which has at least one of its coördinates necessarily imaginary. Thus in projective coördinates with a real triangle and unit point, $(i, 2, 3)$ is imaginary whereas $(i, i, i) \equiv (1, 1, 1)$ is real.

words the real points of the line are defined in absolute coördinates by conjugate complex numbers. There is one real point on this line, *viz.*, the point (2, 2), whose absolute coördinates are real. On the other hand the number pairs (4, 0), (5, −1), (i, $4 - i$), ($3 + \sqrt{2}$, $1 - \sqrt{2}$) satisfy equation (4) but they cannot be represented on the line.[1] Accordingly these pairs are absolute coördinates of imaginary points on the line.

The foregoing discussion suggests the algebraic criterion for reality of points. Indeed, referring to (1) and (2) it is plain that when X, Y are real the point is real and the absolute coördinates are conjugate complex numbers.[2] Conversely if x, $\bar{x}$ are conjugate complex numbers, the point is real and the rectangular coördinates are real. Or

A point is real or imaginary according as its absolute coördinates are or are not conjugate complex numbers.

Parametric equations in absolute coördinates. Heretofore we have thought of t as a real parameter running along a line and taking all values from $-\infty$ to $+\infty$. Now it is more convenient to suppose t to run around a circle of unit radius with center at the origin. In other words t is a complex number of absolute value 1. We do not restrict t to be a root of unity as defined in the previous section,—the only requirement is $|t| = 1$. The properties of t as an operator are extended to apply to all points on the unit circle. An important consequence is that the conjugate of t is $1/t$.[3]

[1] For if we plot the x only, the first, second and fourth points would be real points on the line $x = \bar{x}$ whereas this line cuts (4) only in the point (2, 2). Thus we can plot only those points whose $\bar{x}$-coördinates are conjugates of the x-coördinates, which indicates the role of $\bar{x}$.

[2] We use complex number to include all the numbers of algebra, a real number being self-conjugate. An imaginary number is one that is not real.

[3] This may be proved directly. For let t be $p + iq$, p, q real, $p^2 + q^2 = 1$. Then $(p + iq)\ (p - iq) = p^2 + q^2 = 1$. Hence conjugate complex numbers of absolute value 1 are reciprocals.

The parametric equations of the unit circle in absolute coördinates are

$$x = t, \qquad \overline{x} = 1/t$$

or homogeneously (5)

$$x = t^2, \qquad \overline{x} = 1, \qquad z = t.$$

For manifestly the x-coördinate of any point on the circle is the same as the parameter. The homogeneous reference lines are two tangents, the circular rays x (on I) and $\overline{x}$ (on J), and their chord of contact z ($\mathcal{L}$). Now z cuts the circle in points with parameters 0, ∞ which are thus the parameters of the points I, J. The parameters of the points cut out by the line $x = \overline{x}$ are 1, -1. We have then four real values of the parameter, 0, ∞, ± 1.

Let us now consider the equation

$$x = f(t) \tag{6}$$

where f is a rational function of t. The coefficients may be any complex numbers but we shall suppose that t is a turn, *i. e.*, lies on the unit circle. Equation (6) carries with it the conjugate

$$\overline{x} = \overline{f}(t) \tag{7}$$

obtained by writing for each complex number including t its conjugate. (6) *and* (7) *are then parametric equations of a rational curve in absolute coördinates.*

To plot the graph of such a curve we assign to t various turn values and compute the corresponding values of x. We then plot the x's as in the complex plane, attending however to the criterion for reality. In practice it usually is sufficient to give to t in addition to 0 and ∞ the simpler roots of unity such as ± 1, $\pm i$, $\pm \omega$, $\pm \omega^2$, etc. It is important to note that if we want successive points on the curve we must assign values to t in the order in which they occur on the unit circle.

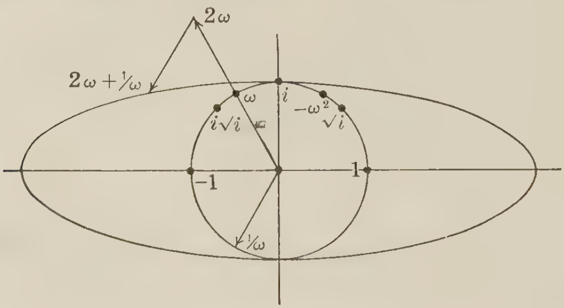
Example. Let us draw the graph of the curve

$$x = 2t + 1/t, \qquad \overline{x} = 2/t + t. \tag{8}$$

Observe first that x and $\overline{x}$ are interchanged when t is replaced by $1/t$ which says that the curve is symmetrical with respect to the line $x = \overline{x}$. Now let t take the values indicated in the table and compute the corresponding values of x.

t	1	$\sqrt{i}$	$-\omega^2$	i	ω	$i\sqrt{i}$	-1
x	3	$2\sqrt{i} - i\sqrt{i}$	$1 - \omega^2$	i	$\omega - 1$	$2i\sqrt{i} - \sqrt{i}$	-3

Plotting the points[1] x they are found to lie on an ellipse.



All that was said about rational curves in Chapter X applies immediately to curves given parametrically in absolute coördinates if we write the equations homogeneously as is always possible. Thus the homogeneous parametric equations of the ellipse (8) are $x = 2t^2 + 1$, $\overline{x} = t^2 + 2$, $z = t$.

The method of absolute coördinates has been developed (under the name of *Vector Analysis* or *Reflexive Geometry*) chiefly by Prof. Morley and his students who have employed it with great success in discussing the metrical properties of curves.[2]

[1] This is best done by a composition of vectors—double the vector for t, then add $1/t$ as in the figure for ω.

[2] See particularly F. Morley, *Transactions*, American Mathematical Society, April, 1900, Jan., 1903, and Jan., 1907.

EXERCISES

1. Any rational power of an nth root of 1 is an nth root.

2. If n is prime every nth root (except 1) is primitive.

3. Among the nth roots one or two are real according as n is odd or even.

4. The primitive nth roots of -1 are primitive $(2n)$th roots of 1.

5. If α is a primitive kth root of 1, $k < n$, then α as an operator generates with respect to multiplication a cyclic subgroup of order k of the G_n of nth roots.

6. The product of the nth roots is 1 or -1 according as n is odd or even. All other elementary symmetric functions of the nth roots are zero. In particular the sum of the nth roots is zero. Prove the last statement graphically.

7. Show that the complex 5th roots of unity can be arranged in sequence such that each number is the square of the preceding.

8. The 7th roots of unity cannot be arranged in a sequence of squares but they may be written in a sequence in which each number is the cube of the preceding.

9. If $\epsilon^i + \epsilon^{-i} = \eta_i$, $\epsilon^n = 1$, show by direct multiplication that $\eta_i\eta_k = \eta_{i-k} + \eta_{i+k}$.

10. If $\epsilon^n = 1$, the number $\epsilon^i + \epsilon^{-i}$ is real. (Prove graphically or show that it is self-conjugate.)

11. As in Ex. 9 let $\epsilon + \epsilon^4 = \eta_1$ and $\epsilon^2 + \epsilon^3 = \eta_2$, $\epsilon^5 = 1$. Show that $\eta_1 + \eta_2 = \eta_1\eta_2 = -1$ and hence the quadratic giving the η's is $\eta^2 + \eta - 1 = 0$. Thence find η_1 and η_2. Also since $\epsilon\epsilon^4 = \epsilon^2\epsilon^3 = 1$ find the values of the ϵ's, thus solving the equation $x^5 - 1 = 0$ by quadratics.

12. Likewise if $\epsilon^7 = 1$, let $\epsilon + \epsilon^6 = \eta_1$, $\epsilon^2 + \epsilon^5 = \eta_2$ and $\epsilon^3 + \epsilon^4 = \eta_3$. Write the cubic equation giving the η's and thus reduce the solution of $x^7 - 1 = 0$ to the solution of a cubic and quadratics.

13. Express algebraically the complete set of nth roots of unity for $n = 9, 12, 16$. How many in each set are primitive? Write the equations giving the primitive roots in each set.

14. An equation $f(x, \bar{x}) = 0$ of the nth degree in absolute coördinates represents a curve of the nth order and conversely.

15. The condition that the two lines

$$a_1x + b_1\bar{x} + c_1 = 0 \qquad\qquad a_2x + b_2\bar{x} + c_2 = 0$$

be parallel (but not coincident) is the same as in Cartesian coördinates viz., $a_1/a_2 = b_1/b_2 \neq c_1/c_2$. The condition that the lines be perpendicular is $a_1/a_2 = -b_1/b_2$.

16. Find the condition that the line $x + \bar{x} = 2k$ cut the circle $x\bar{x} = 1$ in points which are (*a*) real and distinct, (*b*) real and coincident, (*c*) imaginary.

17. The equation $ax^2 + 2hx\bar{x} + b\bar{x}^2 + 2gx + 2f\bar{x} + c = 0$ where the coefficients are real will represent a hyperbola, a parabola, an ellipse, proper or degenerate, according as $ab - h^2 > 0, = 0, < 0$.

18. Plotting x instead of $\bar{x}$ makes the graph correspond to the graph in rectangular coördinates. If we followed the reverse procedure we should get the curve reflected in the line $x - \bar{x} = 0$.

19. Transform the following equations from rectangular to absolute coördinates or the reverse.

(*a*) $XY = k$	(*b*) $(X^2 + Y^2)^2 = a^2(X^2 - Y^2)$
(*c*) $(x^2 + \bar{x}^2)^2 - x\bar{x} = 0$	(*d*) $(x + \bar{x})^4 - x\bar{x} = 0$
(*e*) $x^3 + \bar{x}^3 - x^2\bar{x}^2 = 0$	(*f*) $x^5 + \bar{x}^5 - x^2\bar{x}^2 = 0.$

20. The equation $x = a + bt$ (absolute coördinates) represents a circle with center at $x = a$ and radius $|b|$. (Show that the distance from $x = a$ to $x = a + bt$ is constant and equal to b, remembering that $|t| = 1$.)

21. Eliminate t in the equations of the ellipse (8) §156 and find the single equation in x and $\bar{x}$. Transform the equation to rectangular coördinates.

22. The equation in absolute coördinates $x = at + 1/t$ represents an ellipse for all values of a except $a = 1$ when the equation represents a line segment. What is the length of the segment?

23. The parametric equations of the unit circle in rectangular coördinates and real parameter are

$$X = \cos\theta = \frac{1 - t^2}{1 + t^2} \qquad t = \tan\frac{\theta}{2}.$$
$$Y = \sin\theta = \frac{2t}{1 + t^2}$$

Find the parameters of the points in which the axes cut the circle. What are the parameters of I and J? Find a binary transformation that changes t into a turn by asking that $i, -i, 0$ go respectively into $0, \infty, 1$. Then transform X and Y into absolute coördinates and obtain the standard equations of the circle $x = t$, $\bar{x} = 1/t$. Obtain these equations by first transforming X and Y and then t.

24. Plot the following curves in absolute coördinates as in §156, either by constructing a table of values or by composition of vectors.

(a) $x = 3t + 1/t$
(b) $x = 4t + 1/t$
(c) $x = t - t^2$
(d) $x = 2t - t^2$
(e) $x = 4t - t^2$
(f) $x = \dfrac{1}{(1 + t)^2}$
(g) $x = (1 + t)^2$
(h) $x = (1 + t)^{\frac{1}{2}}$
(i) $x = (1 + t)^{\frac{1}{3}}$
(j) $x = 2t^2 + t - 1/t$
(k) $x = t + 1/t^2$
(l) $x = 2t + 1/t^2$.

25. Write the parametric equations in homogeneous absolute cöordinates of each curve in the preceding exercise.

157. The binary cyclic collineation group.—We have seen (§154, Ex. 1(*a*)) that the collineations in one dimension

$$t' = \frac{at + b}{ct + d} \tag{1}$$

constitute a group. Let us ask under what conditions the group is cyclic. First we shall prove that

A finite binary collineation group cannot contain a parabolic collineation.

For every element of a finite group generates a cyclic group (§154). But a parabolic collineation (§82, (2))

$$t' = t + k \tag{2}$$

cannot generate a group since obviously no power of (2) can be the identity,—unless indeed $k = 0$ but then we should have the trivial case of a group with a single element, 1.

Every element in a cyclic group accordingly will have two and only two distinct fixed points. Hence the generating collineation can be written homogeneously in the canonical form

$$\rho t_1' = k_1 t_1, \qquad \rho t_2' = k_2 t_2. \tag{3}$$

If now (3) is of period n we must have $k_1{}^n = k_2{}^n = 1$, in other words the k's are nth roots of unity. Or dividing

out one of the k's *we may write the generating transformation of the cyclic group of order n in the form*

$$\rho t_1' = \epsilon t_1, \qquad \rho t_2' = t_2 \tag{4}$$

where ϵ *is a primitive nth root of unity* and (§155) may be taken as $e^{\frac{2\pi i}{n}}$ by using the proper power of (4). The non-homogeneous form is

$$t' = \epsilon t. \tag{5}$$

It follows that *there is a single type of binary cyclic group, viz.*, (5).

It appears also that every element in (5) has the same two fixed points and we may say

Two points of a line are fixed under the operations of a binary cyclic group while all other points are arranged in sets of n conjugate points.

If α is the coördinate of a point on the line, the collineations of the group will carry α successively into the points

$$\alpha, \epsilon\alpha, \epsilon^2\alpha \quad . \quad . \quad . \quad \epsilon^{n-1}\alpha.$$

Hence the equation,

$$t^n - \alpha^n = 0 \tag{6}$$

for varying α, *gives the* ∞^1 *sets of conjugate points under the group.*

Cyclic group on the conic. The results of this section are valid not alone for the points of a line (or lines of a pencil) but are equally significant for the geometry on a conic or any rational (point or line) curve. Thus on the conic

$$x_1 = t^2, \qquad x_2 = t, \qquad x_3 = 1 \tag{7}$$

the collineation (5) will generate a cyclic group on the parameter that carries a general point t into a set of n

conjugate points. The only exceptions are the points 0, ∞ which are fixed. All sets of conjugate points are given by equation (6) where of course α is to be interpreted as a parameter.

158. Cyclic collineation groups in the ternary domain. A finite ternary group cannot effect a parabolic collineation on any fixed line in virtue of §§149,157. It follows that

A ternary group which is finite cannot contain a collineation of type III, IV, or V.

For under these types the binary collineation on one fixed line is parabolic (§149, Ex. 2). We are left then with types I and II both of which may occur.

Referring to the canonical form of the general collineation (§147, (3)) we have at once, a cyclic group being finite:

The cyclic groups in the ternary domain can be generated by the collineation

$$\rho x_1' = \alpha_1 x_1, \quad \rho x_2' = \alpha_2 x_2, \quad \rho x_3' = \alpha_3 x_3 \tag{1}$$

where, because of the periodicity requirement, *the α's are roots of unity.*

While in the binary domain there is a single type of cyclic group, we recognize two general classes of ternary cyclic groups:

(a) those with a single fixed (invariant) triangle

(b) the homologies with a center of fixed lines and an axis of fixed points, *i. e.*, with ∞^2 fixed triangles.

If (1) is of order n the multipliers α must be nth roots of unity at least one of which is primitive. By making use of the factor of proportionality we may take one of the multipliers say $\alpha_3 = 1$. Then if ϵ is a primitive nth root of unity say $e^{\frac{2\pi i}{n}}$ the others may be written $\alpha_1 = \epsilon^r$, $\alpha_2 = \epsilon^s$ and the collineation becomes

$$\rho x_1' = \epsilon^r x_1, \quad \rho x_2' = \epsilon^s x_2, \quad \rho x_3' = x_3 \tag{2}$$

where one multiplier say ϵ^r must be a primitive nth root. Then there is some positive integer m such that $(\epsilon^r)^m = \epsilon$ (§155). Hence taking the mth power of (2) as the generating transformation of the group we have as the *canonical form of the generating transformation of the cyclic group with an invariant triangle:*

$$x_1' = \epsilon x_1, \qquad x_2' = \epsilon^k x_2, \qquad x_3' = x_3, \qquad \epsilon = e^{\frac{2\pi i}{n}}, \quad k \neq 1. \tag{3}$$

Likewise we may take the multipliers of a homology to be proportional to ϵ, 1, 1 (or 1, ϵ, ϵ) since two of them are equal. Accordingly the generating transformation of the cyclic group consisting of homologies may be written

$$x_1' = \epsilon x_1, \quad x_2' = x_2, \quad x_3' = x_3. \tag{4}$$

The groups of type (a) may be further subdivided into species projectively distinct when $n > 5$. An important special case is that generated by the *orthogonal* transformation, *i. e.*, one with two reciprocal multipliers. Thus we may take as the *canonical form of the orthogonal collineation*

$$x_1' = \epsilon x_1, \quad x_2' = \epsilon^{-1} x_2, \quad x_3' = x_3. \tag{5}$$

A reflexion is manifestly both an orthogonal transformation and a homology.

Consider now the effect on the conic (7) §157 of the collineation

$$t' = \epsilon t. \tag{6}$$

If the coördinates x_i of a point are transformed (by (6)) into x_i' we have

$$\begin{aligned} x_1' &= t'^2 = \epsilon^2 t = \epsilon^2 x_1 \\ x_2' &= t' \;\, = \epsilon t \;\,= \epsilon x_2 \\ x_3' &= 1 \;\;\, = 1 \;\;\, = x_3 \end{aligned} \tag{7}$$

or, dividing by ϵ since only the ratios are essential

$$x_1' = \epsilon x_1, \quad x_2' = x_2, \quad x_3' = \epsilon^{-1} x_3, \tag{8}$$

an orthogonal collineation which evidently leaves unaltered the ternary equation of the conic

$$x_2^2 - x_3x_1 = 0. \tag{9}$$

Hence *the binary collineation* (6) *induces the* (*orthogonal*) *ternary collineation* (8) *which generates a ternary cyclic* G_n *under which the conic is invariant.*

Thus to the binary cyclic group which permutes the points (parameters) t of the conic corresponds a ternary cyclic group of the same order which permutes the points x of the curve. The only fixed elements of the binary group are the points $t = 0, \infty$ while the ternary group has an additional fixed point,—the intersection of the tangents at these points.

EXERCISES

1. Apart from the homology there is a single type of ternary cyclic G_5, that generated by the orthogonal collineation. (The only possible generating transformations are those with multipliers proportional to $1, \epsilon, \epsilon^2$; $1, \epsilon, \epsilon^3$ and $1, \epsilon, \epsilon^4$, $\epsilon^5 = 1$. Now the first set and the square of the second set when divided by ϵ reduce to the third set (although in a different order).

2. There are two projectively distinct types of ternary cyclic G_7's in addition to the homology. Find the generating transformations and show that one has a pencil of (proper) fixed conics but the other has no proper fixed conic.

3. Find the projectively distinct types of ternary cyclic G_6's.

4. A ternary cyclic G_n, n even contains at least one reflexion. Hence any invariant curve may be metrically placed so that it will have (a) a line of symmetry or (b) a center (§144, end).

5. A ternary orthogonal collineation of period 3 has three pencils of invariant conics. Find them. (If the whole conic is multiplied by ω or ω^2 the conic is invariant.)

6. Find the condition that the binary collineation S: $t_1' = at_1 + bt_2$, $t_2' = ct_1 + dt_2$ be of period 3, *i. e.*, that $S^3 = 1$. (We can write the condition $S^2 = S^{-1}$. Now ask that the coefficients of S^2 be proportional to those of S^{-1} and find a relation among them. Or write

a collineation of period 3 in canonical form and find a relation,—homogeneous in the coefficients,—connecting the invariants.)

Ans. $I_1^2 - I_2 = 0$.

7. Find the condition that S, Ex. 6 be of period 4, *i. e.*, $S^4 = 1$, or $S^2 = S^{-2}$. *Ans.* $I_1^2 - 2I_2 = 0$.

8. Show that the following rational curves are invariant under cyclic groups of the orders indicated.

$$(a)\quad x_1 = t^4,\ x_2 = t,\ x_3 = t^3 + 1$$
$$x_2(x_1 + x_2)^3 - x_1x_3^3 = 0 \qquad (G_3)$$
$$(b)\quad x_1 = t^4,\ x_2 = t^2,\ x_3 = t^3 + 1$$
$$x_2(x_1^3 - x_2^3 + 2x_1x_2x_3) - x_1^2x_3^2 = 0 \qquad (G_3)$$
$$(c)\quad x_1 = t,\ x_2 = t^2,\ x_3 = t^4 + 1$$
$$x_1^4 + x_2^4 - x_1^2x_2x_3 = 0 \qquad (G_4)$$
$$(d)\quad x_1 = t,\ x_2 = t^5,\ x_3 = t^4 + 1$$
$$x_1(x_1 + x_2)^4 - x_2x_3^4 = 0 \qquad (G_4)$$
$$(e)\quad x_1 = t,\ x_2 = t^2,\ x_3 = t^5 + 1$$
$$x_1^5 + x_2^5 - x_1^3x_2x_3 = 0 \qquad (G_5)$$
$$(f)\quad x_1 = t^3,\ x_2 = t,\ x_3 = t^5 + 1$$
$$x_1^5 - x_2^5 + x_1x_2^2x_3^2 - 2x_1^3x_2x_3 = 0 \qquad (G_5)$$
$$(g)\quad x_1 = t,\ x_2 = t^{n+1},\ x_3 = t^n + 1$$
$$x_1(x_1 + x_2)^n - x_2x_3^n = 0. \qquad (G_n)$$

9. In (*a*) Ex. 8 the three parameters given by $t^3 + 1 = 0$ are all attached to the same point, *i. e.*, they are the parameters of a *triple* point. Likewise curve (*d*) has a 4-fold point and (*g*) an n-fold point.

159. Binary and ternary dihedral collineation groups.— We saw in the preceding section that there is an intimate connection between binary and ternary cyclic groups with an invariant conic. In this section we shall extend this discussion to a second important class of groups. Since we are dealing with parallel theories some conventions in the terminology and notation are desirable. A binary group of order n, which of course refers to the parameter, we shall denote by g_n. The corresponding ternary group we shall designate by G_n. A collineation of period two in the binary domain we shall call an involution, in the ternary domain a reflexion.

We shall treat first the unit circle whose equations in absolute coördinates are (§156)

$$x = t, \ \overline{x} = 1/t, \text{ or } x\overline{x} = 1. \tag{1}$$

The center of the circle which corresponds to the Cartesian origin we shall sometimes call the *origin.*

The involution

$$t' = 1/t \tag{2}$$

induces the reflexion

$$x' = \overline{x}, \quad \overline{x}' = x \tag{3}$$

which leaves the circle unaltered. Likewise the binary collineation

$$t' = \epsilon t \tag{4}$$

determines the ternary collineation

$$x' = \epsilon x, \quad \overline{x}' = \epsilon^{-1}\overline{x} \tag{5}$$

which also leaves the circle invariant. Obviously the effect of (5) is to rotate the point x about the center through the angle $2\pi/n$.

Thus the two binary collineations (2) and (4) generate a group g_{2n} containing n involutions and n collineations, including the identity, of period n. Similarly the ternary collineations (3) and (5) generate a G_{2n} consisting of n reflexions and n rotations, including the identity, of period n. These are the *dihedral*[1] collineation groups in one and two dimensions.

The transformations of the cyclic subgroup binary and ternary are

$$t' = \epsilon^k t \qquad \begin{aligned} x' &= \epsilon^k x \\ \overline{x}' &= \epsilon^{-k}\overline{x}, \end{aligned} \qquad k = 1, 2, \ldots n. \tag{6}$$

[1] So called because its invariant figure is a *dihedron*,—the limiting form of a regular polyhedron,—consisting of but two faces which are coincident regular polygons of n vertices. The cyclic G_n rotates the dihedron into itself about the center. And a reflexion may be regarded as a rotation (in space) of the dihedron into itself about an axis of symmetry, *i. e.*, a line joining a vertex to the center, or when n is even a line connecting the midpoints of opposite sides. See figures, p. 338.

And the involutions and reflexions are

$$t' = \epsilon^k/t \qquad \begin{aligned} x' &= \epsilon^k \overline{x} \\ \overline{x}' &= \epsilon^{-k} x. \end{aligned} \tag{7}$$

Some properties of the ternary group. In the metrical form here assumed an axis of reflexion is a line of symmetry and the corresponding center is the zenith of the axis (§144). Hence

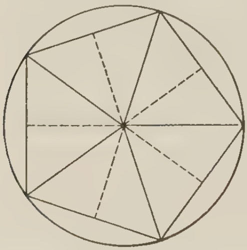
The n axes of reflexion pass through a point, the origin, while the centers of reflexion lie on a line, the line at infinity.

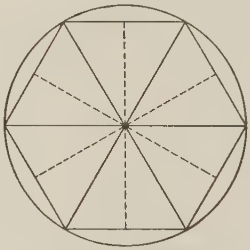
Now the double points of an involution lie on the axis of the associated reflexion. The double points of the involution (7) which are given by $t^2 = \epsilon^k$ are obviously cut out by the line $x - \epsilon^k \overline{x} = 0$. Therefore

The axes of reflexion $x^n - \overline{x}^n = 0$ are n equispaced lines about the origin.

There is however an important distinction between dihedral groups of order $2n$ according as n is odd or even. It will suffice to point out this distinction for the ternary group.

Case I. n odd. The application of the cyclic G_n to any axis of reflexion say $x - \overline{x} = 0$ carries it into each of the





others, *i. e., the axes are conjugate under the G_{2n}.* (See figure left.)

Case II. n even. When n is even the line $x - \overline{x} = 0$ assumes only $n/2$ distinct positions under the group (see

figure, right.) Likewise the axis $x - \epsilon\bar{x} = 0$ under the rotations of G_n coincides in turn with the remaining $n/2$ axes. Thus the axes separate into two sets of $n/2$ lines which are not conjugate to each other. If $n = 2m$ the two sets of axes are

$$x^m - \bar{x}^m = 0, \; x^m + \bar{x}^m = 0. \tag{8}$$

Thus each set is equispaced within itself, the angle between successive lines being π/m, and the two together make up n lines separated at intervals of π/n. It follows that *the centers of reflexion corresponding to one of the conjugate sets of $n/2$ axes lie on the lines forming the other.*

Again the rotation $x' = \epsilon^m\bar{x}$ is of period two. Hence *the group has an extra reflexion with the origin as center and the line at infinity as axis.*

Special sets of conjugate points. In general the points of the circle (both as binary and ternary elements) are arranged in sets of $2n$ conjugate points under the operations of the groups. Let us now examine the groups for conjugate sets containing fewer than $2n$ points. Such sets must be fixed under some transformations of the group, *i. e.*, under a subgroup. First the cyclic g_n leaves each of the points 0, ∞ unaltered while the n involutions merely interchange them. Thus the parameters of the circular points as a pair are invariant under the entire group. Again every involution has two fixed points each of which is carried by the cyclic g_n into n distinct points. Accordingly if we operate on -1 and $+1$ with g_n we obtain the two special sets of n conjugate points $t^n + 1 = 0$ and $t^n - 1 = 0$. All other points of the circle belong to general sets of $2n$ conjugate points. The equation representing the general set will be of degree $2n$ and must be invariant under the generating transformations of the group. It may therefore be written

$$at^{2n} + bt^n + a = 0 \quad \text{or} \quad t^{2n} + kt^n + 1 = 0.$$

The special sets of the binary group all lie on the circle but those of the ternary group are not so limited since the ternary group transforms the whole plane. We note first a special set consisting of a single point, the origin, for the point is at once the center of rotation and the intersection of the axes of reflexion. Next the cyclic subgroup G_n leaves each of the circular points stationary while each reflexion interchanges the two. That is the circular points (as a pair), and consequently the line at infinity, are invariant under G_{2n}. Again each center of reflexion is a fixed point of the reflexion, hence the n centers constitute a special set of n points lying on a line. Finally a point on an axis is fixed under the corresponding reflexion, accordingly such a point belongs to a set of n conjugate points. We get thus ∞^1 special sets of n points, one point of each set lying on each axis. This concludes the enumeration of special sets which we tabulate below.

Sets of Conjugate Points of Dihedral Groups

Binary g_{2n}	*Ternary* G_{2n}
1° One pair of points $2t = 0$.	1° One point, the intersection of the axes.
2° Two special sets of n points $t^n + 1 = 0,\ t^n - 1 = 0$.	2° A pair of points, the absolute.
3° ∞^1 general sets of $2n$ conjugate points $t^{2n} + kt^n + 1 = 0$.	3° One set of n points, the centers of reflexion.
	4° ∞^1 sets of n points, lying on the axes of reflexion.
	5° ∞^2 general sets of $2n$ points.

The position of a general set of conjugate points of G_{2n} is readily found. For the cyclic G_n, operating on a point x and its image $\overline{x}$ in the axis $x - \overline{x} = 0$, obviously carries

each point into n points at equal intervals on a circle with center at the origin and radius $= |x|$. Or

Every member of the pencil of concentric circles (double contact conics)

$$x\bar{x} = k^2 \tag{9}$$

is invariant under the dihedral G_{2n}. *Each of these circles contains* ∞^1 *general sets of* $2n$ *conjugate points which lie at the vertices of two regular inscribed polygons.*

Any curve which is invariant under the dihedral G_{2n} *of this section possesses n-fold symmetry since it is symmetrical about each of n equispaced lines (axes of reflexion) through the origin.* Conversely, *any curve which is symmetrical about each of n equispaced lines on a point admits a dihedral* G_{2n}. When n is odd the symmetry of the curve is the same with respect to each axis, but when n is even the symmetry about one half the axes differs from the symmetry about the other half. When n is even the curve is also symmetrical about the origin, *i. e.*, the origin is a *center* of the curve.

160. The projective properties of dihedral groups.— There is no difficulty in translating the principal results of the preceding section into projective language. We may either suppose the whole plane projected or we may interpret the variables as projective coördinates. In either case the circle is replaced by the conic

$$x_1 = t^2, \quad x_2 = 1, \quad x_3 = t \tag{1}$$

where the x's are projective coördinates and t is a real parameter. Practically all of the remarks concerning the binary group are still valid. In particular the binary equations, including those which represent conjugate sets, are identical.

The generating transformations of the ternary group are the reflexion

$$x_1' = x_2, \quad x_2' = x_1, \quad x_3' = x_3 \tag{2}$$

and the orthogonal collineation of period n

$$x_1' = \epsilon x_1, \quad x_2' = \epsilon^{-1} x_2, \quad x_3' = x_3, \tag{3}$$

which of course is no longer a rotation.

The G_{2n} thus consists of n reflexions and n transformations of period n. The axes of reflexion are on a point and the centers are on a line, point and line being pole and polar with respect to the invariant conic. This point and line are fixed elements of the group and form part of the fixed triangle of the group,—the triangle of reference.

When n is odd the axes of reflexion are all conjugate, but when n is even they break up into two sets of $n/2$ such that the lines of each set are conjugate but the two sets are not conjugate. The equations of the axes are

$$x_1^n - x_2^n = 0, \text{ or if } n = 2m, (x_1^m - x_2^m)(x_1^m + x_2^m) = 0. \tag{4}$$

When n is even the centers of reflexion corresponding to one set of $n/2$ axes lie on the lines of the other set. Again if n is even the ternary group has an extra reflexion with $u_3 = 0$ for center and $x_3 = 0$ for axis.

The sets of conjugate points of G_{2n} are the same as those listed in §159 except 2° which is replaced by a pair of real points, the vertices u_1 and u_2 of the reference triangle.

The G_{2n} leaves unaltered every member of the pencil of double contact conics

$$x_1 x_2 + \lambda x_3^2 = 0. \tag{5}$$

But these are not the only rational curves which admit dihedral groups. In fact there is always at least one type of rational curve of order n invariant under a dihedral G_{2n}.[1] The equations which give the sets of conjugate parameters are the same as those for the conic, §159, table. The special sets of points on the curve,—points t as well as points x,—are cut out by the axes of reflexion, with the possible exception of one set which may fall at the centers

[1] See footnote 2, p. 343.

of reflexion when these are on the curve. The general sets of conjugate points are cut out by the pencil of conics (5).

A curve which is invariant under a collineation group is called *self-projective*. The projectively distinct types of self-projective plane curves of the lower orders have been enumerated by various writers.[1] The types of rational curves through the seventh order have been tabulated by the author.[2]

The abstract dihedral group. Any group which has the same abstract structure as that considered in this section is called a dihedral group. Thus, as is readily verified by referring to §159, we may define the abstract dihedral group as one which is generated by an element S of order n and an element T of order 2 such that T transforms S into its inverse. That is, S and T satisfy the relations

$$S^n = T^2 = 1, \quad TST^{-1} = S^{-1}. \tag{6}$$

A knowledge of dihedral collineation groups is essential to an understanding of the other finite collineation groups in the binary and ternary domain since nearly all such groups contain dihedral subgroups.[3] Indeed dihedral groups in the abstract occur as subgroups in practically all important

[1] Quartics by Wiman, *Svensk, Akad. Bihang*, Vol. 21 (1895) and Ciani, *Rendiconti, Istituto Lombardo*, Series 2, Vol. 33 (1900); Quintics, by Synder, *American Journal of Mathematics*, Vol. 30 (1908) and Ciani; Sextics, Tappan, *American Journal of Mathematics*, Vol. 37 (1915). A summary for the quartic and quintic is given in Loria, *Spezielle Ebene Kurven*, second edition, Vol. 1, p. 107, 249.

[2] Self-Projective Rational Curves of the Fourth and Fifth Orders, *American Journal*, Vol. 36 (1914) and Self-Projective Rational Sextics, *Ibid.*, Vol. 38 (1916). The results for the rational septimic are not yet published. In particular the equations of the rational curves of order n which admit cyclic G_n's and dihedral G_{2n}'s are given, first paper, p. 66.

[3] For references to the literature as well as a discussion see Blichfeldt, *Finite Collineation Groups* or Part II, Miller, Blichfeldt, Dickson, *Finite Groups*. An excellent account of finite binary collineation groups is given by Klein, who first determined the five types, in his *Vorlesungen über das Ikosaeder*, Leipzig, 1884.

types of finite groups. All finite groups of course contain cyclic subgroups (§154).

EXERCISES

1. If S and T are generating elements of a dihedral G_{2n} so that $S^n = T^2 = 1$ and $TST^{-1} = S^{-1}$ show that the elements of the group can be arranged in the two sets

$$\begin{array}{llll} 1, & S, & S^2, & \ldots\; S^{n-1} \\ T, & TS, & TS^2, & \ldots\; TS^{n-1} \end{array}$$

(Either make use of the properties of §150 or prove geometrically with the dihedral group in metrical form.)

2. Show that the dihedral group can be generated by two elements of period two, *e. g.*, T and TS above whose product is of period n. Any two elements of period 2 of a dihedral group whose product is of order k will generate a dihedral subgroup of order $2k$.

3. Enumerate the subgroups of the dihedral G_6, and G_8.

4. If R_1, R_2, R_3 are the elements of period two in the dihedral G_6, show geometrically or otherwise that $R_1R_2R_3 = R_2$ and hence $(R_1R_2R_3)^2 = 1$. By operating on a point successively with R_1, R_2, R_3, R_1, R_2, R_3 we obtain a general set of 6 conjugate points which lie on an invariant conic of G_6. These points form a Pascal hexagon whose opposite sides meet at the centers of reflexion. The axes of reflexion join opposite vertices of the corresponding Brianchon hexagon and meet at the Brianchon point. Construct the inscribed hexagon when R_1, R_2, R_3 are reflexions in the lines

$$x - \overline{x} = 0, \qquad x - \omega\overline{x} = 0, \qquad x - \omega^2\overline{x} = 0.$$

5. If R_i denote the n elements of period two in a dihedral G_{2n} (except the extra one for even n), then $(R_1R_2R_3 \;.\;.\;.\; R_n)^2 = 1$ and the product of the R's is a reflexion. A general set of conjugate points is constructed by operating on a point successively with R_1, R_2, . . . R_n, R_1, R_2, . . . R_n.

6. If a_1 and a_2 are the centers of two reflexions R_1 and R_2 that send a conic into itself (§144) the product R_1R_2 has in general two fixed points on the conic,—cut out by the line a_1a_2. (What is the exceptional case?) These fixed points are the double points of the involution determined by the axes of reflexion.

7. Show that the following rational curves are invariant under dihedral groups, binary and ternary, as indicated.

(*a*) $x_1 = t^3 + 1,\ x_2 = t^4 + t,\ x_3 = t^2$

$$x_3(x_1^3 + x_2^3) - x_1^2x_2^2 = 0 \qquad (G_6)$$

(*b*) $x_1 = t,\ x_2 = t^3,\ x_3 = t^4 + 1$

$$(x_1^2 + x_2^2)^2 - x_1x_2x_3^2 = 0 \qquad (G_8)$$

(*c*) $x_1 = t^3,\ x_2 = t^2,\ x_3 = t^5 + 1$

$$x_1^5 + x_2^5 - x_1^2x_2^2x_3 = 0 \qquad (G_{10})$$

(*d*) $x_1 = t,\ x_2 = t^4,\ x_3 = t^5 + 1$

$$x_1^5 + x_2^5 + x_1x_2x_3(3x_1x_2 - x_3^2) = 0 \qquad (G_{10})$$

(*e*) $x_1 = t,\ x_2 = t^5,\ x_3 = t^6 + 1$

$$(x_1^3 - x_2^3)^2 + x_1x_2x_3^2(4x_1x_2 - x_3^2) = 0. \qquad (G_{12})$$

8. Curve (*a*), Ex. 7, has a triple point whose parameters are $t^3 + 1 = 0$. Curve (*b*) has two double points on $x_3 = 0$.

9. Write the equation of a tangent line of curve (*c*). Show that the tangent at the point $t = -1$ cuts in *three* coincident points, *i. e.*, it is an inflexional tangent. Since a point of inflexion must be transformed into a point of inflexion, $t^5 + 1 = 0$ gives five inflexions. These inflexions lie on a line and are the centers of the five reflexions of the group. Find a conic of the invariant pencil $x_1x_2 = kx_3^2$ which has five contacts with the curve. (These contacts must be a special set of five conjugate points.)

10. Repeat Ex. 9 for curve (*d*) in Ex. 7.

11. Find a conic of the invariant pencil that has six contacts with curve (*e*) Ex. 7.

12. Write the equations of each curve in Ex. 7 in absolute coördinates by the substitutions $x_1/x_3 = x,\ x_2/x_3 = \bar{x}$. Plot each of the curves in absolute coördinates, attending to the n-fold symmetry.

13. Curve (*c*) Ex. 7, when written in absolute coördinates, cuts the unit circle at the 15th roots of -1 exclusive of the 5th roots. The parameters on the quintic of these points are the 15th roots of 1 exclusive of the 5th roots.

14. Consider the four collineations

$$\begin{array}{lcccc} & R_1 & R_2 & S & T \\ x_1' = & -x_1 & x_1 & x_3 & x_1 \\ x_2' = & x_2 & -x_2 & x_1 & x_3 \\ x_3' = & x_3 & x_3 & x_2 & x_2. \end{array}$$

R_1 and R_2 generate a dihedral G_4 and with S a G_{12}. The four collineations generate a G_{24} which contains the G_4 and G_{12} as subgroups and

which leaves the four points $(1, \pm 1, \pm 1)$ (as a set) and the conic $x_1^2 + x_2^2 + x_3^2 = 0$ invariant. Write the complete set of transformations constituting the G_{12} and the G_{24}. The whole group is an octahedral G_{24} isomorphic with the permutation group of four things (§154, Ex. 8) since it permutes the points in all possible ways.

The next five exercises are taken from a paper by L. E. Wear, *American Journal of Mathematics*, 1920, p. 113 ff.

15. The two conics $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = 0$ and $b_1x_1^2 + b_2x_2^2 + b_3x_3^2 = 0$ are separately invariant under the dihedral G_4, Ex. 14. The polarity $\Pi: u_1 = \sqrt{a_1b_1}x_1$, $u_2 = \sqrt{a_2b_2}x_2$, $u_3 = \sqrt{a_3b_3}x_3$ interchanges the two conics. The products of Π by the collineations of G_4 give three other polarities that interchange the two conics. The four polarities and the four collineations constitute a group G_8 that leaves the conics as a pair invariant. Find the base conics of the four polarities.

16. The transformations R_1, R_2 and T of Ex. 14 generate a collineation G_8. The products of the elements of the G_8 and the polarity $\Pi: u_1 = ax_1$, $u_2 = \sqrt{bc}x_3$, $u_3 = \sqrt{bc}x_2$ are eight correlations which form with G_8 a G_{16} which leaves invariant (as a pair) the two conics $ax_1^2 + bx_2^2 + cx_3^2 = 0$, $ax_1^2 + cx_2^2 + bx_3^2 = 0$. Write out the transformations of the group and show that six of the correlations are polarities and the other two are of period 4.

17. The two conics (as a pair)

$$x_1^2 + \omega x_2^2 + \omega^2 x_3^2 = 0, \quad x_1^2 + \omega^2 x_2^2 + \omega x_3^2 = 0$$

are invariant under a G_{48} consisting of the 24 collineations in the G_{24} of Ex. 14 and 24 correlations,—the products of the collineations in G_{24} by the polarity $u_1 = x_1$, $u_2 = x_3$, $u_3 = x_2$. Ten of the correlations are polarities and the rest are of periods 3 and 4.

18. The base conics of two of the polarities in Ex. 16 are $ax_1^2 \pm 2\sqrt{bc}x_2x_3 = 0$. These conics are mutually autopolar (Ex. 14, §145). Further each conic and either conic of Ex. 16 are a mutually autopolar pair.

19. Find the invariant relation (§§141–2) between the two conics of Ex. 16. (Such a relation must be homogeneous and involve the coefficients of the conics to the same degree.) *Ans.* $\Delta\Theta'^3 = \Delta'\Theta^3$.

20. Find the invariant relations on the conics in Ex. 17.

21. The binary collineations

$$R_1: t' = -t, \quad R_2: t' = \frac{1}{t}, \quad S: t' = \frac{t + i}{t - i}, \quad i = \sqrt{-1}$$

of which R_1 and R_2 are involutions, while S is of period 3, generate a g_{12}, the tetrahedral group in the binary domain (Ex. 11, §154). Write the entire set of 12 collineations. The group contains three involutions which with the identity form a dihedral subgroup g_4. The remaining elements of the group are of period 3 and can be arranged into four cyclic subgroups g_3. Find the double points of the involutions and the fixed points of the cyclic g_3's. The first which correspond to the middle points of the edges of the tetrahedron are a special set of 6 conjugate points. And the second are two special sets of 4 conjugate points which correspond to the vertices and the middle points of the faces.

22. The transformations R_2 and S, Ex. 21 and $T: t' = it$, generate a binary collineation g_{24} which contains the tetrahedral g_{12} as an invariant subgroup. Write the additional elements of the enlarged group which is the octahedral group. Since T is of period 4 it represents a rotation of the octahedron about an axis joining two opposite vertices. Hence the fixed points of T, *viz.*, 0, ∞ will correspond to two vertices of the octahedron. Since a vertex must go into a vertex there must be a special set of six conjugate points. These are the fixed points of the three cyclic g_4's and are found by operating on the fixed points of T by the other elements of the group. The six points are thus found to be $t(t^4 - 1) = 0$. The group contains two other special sets, the midpoints of the faces or *cube vertices* and the midpoints of the edges. Find these special sets. (The first must be fixed points of the cyclic g_3's while the others are double points of involutions.) *Ans.* $t^8 + 14t^4 + 1 = 0$ and $t^{12} - 33t^8 - 33t^4 + 1 = 0$.

23. The octahedral group leaves unaltered the projective lemniscate, Ex. 7, (*b*).

161. A metrical aspect of collineations.—We have seen (§154, Ex. 1(*b*)) that the totality of collineations in the plane represented by equations (1) §146 constitute an 8-parameter group which we may call the *general projective group*. Since the collineations of this group are simply analytic expressions for projections we may say that projective geometry comprises those properties which are invariant under the projective group.[1] This accounts for the importance of invariants in the projective geometry of curves.

[1] *Cf.* §36.

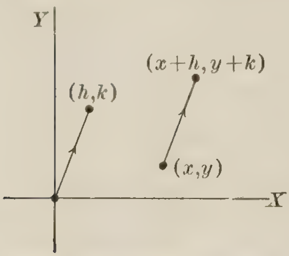
In Cartesian coördinates the general collineation may be written homogeneously

$$\begin{aligned} x' &= a_1x + b_1y + c_1z \\ y' &= a_2x + b_2y + c_2z \\ z' &= a_3x + b_3y + c_3z \end{aligned} \tag{1}$$

or non-homogeneously

$$\begin{aligned} x' &= \frac{a_1x + b_1y + c_1z}{a_3x + b_3y + c_3z} \\ y' &= \frac{a_2x + b_2y + c_2z}{a_3x + b_3y + c_3z}. \end{aligned} \tag{2}$$

Now any point whose coördinates reduce the denominator of (2) to zero, *i. e.*, any point on the line (a_3, b_3, c_3) will be transformed into a point at infinity. In other words (2) may be regarded as *the projection that sends the line* $a_3x + b_3y + c_3z = 0$ *into the line at infinity.*[1]



A very special case of (2) is the transformation

$$x' = x + h, \qquad y' = y + k \tag{3}$$

which transfers the point (0, 0) to the point (h, k), and carries every other finite point of the plane the same distance in the same direction. The collineation (3) is called a *translation*[2] since the same effect would be accomplished by sliding the plane over itself without turning.

A translation evidently leaves unaltered every line of the parallel pencil running in the direction of translation.

[1] This furnishes an analytic justification for postulating (a) points at infinity which (b) lie on a line. For without (a) we should have a line of points without corresponding points and without (b) we should have one line which did not correspond to a line.

[2] The student must distinguish this from the transformation of coördinates in elementary geometry which is a change of the reference scheme. Here the *points* themselves and not the axes are moved.

It also leaves unaltered every point on the line at infinity. For writing (3) homogeneously thus

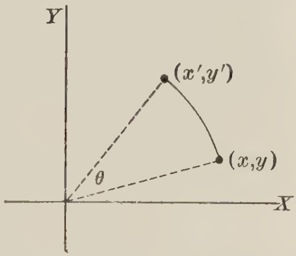
$$x' = x + hz, \quad y' = y + kz, \quad z' = z$$

we have when $z = 0$, $x' = x$, $y' = y$, *i. e.*, the collineation on the line at infinity is the identity. Or

A translation is an elation with the line at infinity as axis and with center at infinity in the direction of translation.

Again the collineation

$$\begin{aligned} x' &= x \cos \theta - y \sin \theta \\ y' &= x \sin \theta + y \cos \theta \end{aligned} \qquad (4)$$



represents a *rotation* of the plane about the origin through the positive angle θ. The rotation leaves unaltered each of the circular points in addition to the origin (§159). It also has three fixed lines, the circular rays from the origin and the line at infinity.

The two collineations (3) and (4) generate a 3-parameter group whose general transformation is

$$\begin{aligned} x' &= x \cos \theta - y \sin \theta + h \\ y' &= x \sin \theta + y \cos \theta + k \end{aligned} \qquad (5)$$

and which is therefore a subgroup of the general projective group (2).

It is geometrically evident that both generating transformations and consequently all elements of the group preserve the distance between two points and the angle formed by two lines. The group is accordingly sometimes called the *congurence group*. It is also called the *group of displacements*, each element being known as a displacement.

A fundamental requirement of Euclid's geometry is that a figure may be freely moved without changing either its size or its shape. But what is meant by "motion"? It is commonly described as *rigid* motion but that only gives the difficulty another name. Since a displacement conforms to Euclid's requirement we may regard rigid motion and displacement as identical when either is *analytically defined* by equations (5). An alternative name for the group of displacements is thus the *group of rigid motion.*

The group of displacements characterizes Euclidean metric geometry exactly as the general projective group characterizes pure projective geometry. Thus Euclidean geometry consists of those properties which are invariant under displacement. When Euclid (by implication) postulated free mobility he prescribed the bounds as well as the essential structure of his geometry.

A basic distinction between projective and Euclidean geometry may be emphasized once more. The line at infinity is not exceptional in projective geometry proper since any line can be projected into any other. But the circular points (the absolute) are invariant under the group of displacements since they are invariant under the generating transformations. Hence the line at infinity is isolated,—a *singular* line under displacement which accounts for the commanding position of the absolute in metric geometry.

For an account of the more important geometries associated with various subgroups of the general projective group the student may consult Veblen and Young, *Projective Geometry*, Vol. II.

EXERCISES

1. The product of two translations is commutative. Likewise the product of two rotations about the same point is commutative. But the product of a translation and a rotation is not commutative.

2. All translations of the plane form an Abelian group.

3. If R_1 and R_2 are two reflexions (in the metrical sense) the product R_1R_2 is a rotation about the intersection of the axes through twice the angle formed by the axes. R_2R_1 is a rotation in the opposite direction. A rotation through the angle π is a product of reflexions in perpendicular lines.

4. If the axes of reflexion in Ex. 3 are parallel, R_1R_2 represents a translation through twice the distance between the lines.

5. Any line of a plane figure can be projected into the line at infinity while at the same time any three lines (no two meeting on the line which is sent into $\mathcal{L}$ and not concurrent) are projected into any three other lines.

6. Any simple quadrangle can be projected into a square.

7. Find a transformation that sends the line $x + y - 2 = 0$ into the line at infinity and which leaves the coördinate axes fixed in position.

8. Any two points can be projected into the circular points while at the same time any other two independent points are projected into assigned positions.

9. The transformation

$$\begin{aligned} x' &= a_1x + b_1y + c_1z \\ y' &= a_2x + b_2y + c_2z \\ z' &= z \end{aligned} \quad \text{or} \quad \begin{aligned} x' &= a_1x + b_1y + c_1, \\ y' &= a_2x + b_2y + c_2, \end{aligned} \quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$$

is called the *affine* transformation. All affine transformations constitute a group, the *affine group*, which is a subgroup of the general projective group. *Affine geometry* consists of those properties which are invariant under the affine group.

10. The affine transformation leaves unaltered $\mathcal{L}$. Hence it transforms parallel lines into parallel lines.

11. The center of a conic under the affine transformation goes into the center of the transformed conic since the center is the polar of the line at infinity. Likewise the midpoint of a segment goes into the midpoint of the transformed segment.

12. The affine transformation has two fixed points on $\mathcal{L}$ and one finite fixed point. Find these fixed points.

13. Taking the finite fixed point as origin the affine transformation can be written

$$\begin{aligned} x' &= a_1x + b_1y \\ y' &= a_2x + b_2y \\ (z' &= z). \end{aligned}$$

14. The affine transformation alters both distances and angles (Use the transformation in Ex. 13.)

15. The finite fixed point is the intersection of two fixed lines. Taking these lines as (oblique) axes the affine transformation is reduced to the canonical form

$$x' = ax, \quad y' = by,$$

where, however, x and y are oblique coördinates.

16. The conditions that the transformation in Ex. 13 send perpendicular lines into perpendicular lines are

$$a_1a_2 = -b_1b_2,\ a_1^2 + b_1^2 = a_2^2 + b_2^2, \text{ or } a_1 = b_2,\ a_2 = -b_1$$

and the transformation becomes

$$x' = ax - by, \quad y' = bx + ay.$$

This is the *equiform* transformation.

17. Show that the equiform transformation alters all distances but preserves all angular magnitudes. Therefore it sends similar figures into similar figures. (Hence the name.)

18. All equiform transformations form a group which is a subgroup of the affine group. The associated geometry is called *equiform geometry.*

19. The equiform group leaves the absolute unaltered.

20. If the equiform transformation is restricted by the requirement that it preserve distances we must have $a^2 + b^2 = 1$. The transformation then becomes the rotation (4), §161.

21. The group of rigid motions is a subgroup of the affine group. It is also a subgroup of the equiform group. We are thus led by successive restrictions from projective geometry through affine geometry and equiform geometry to Euclidean metric (congruence) geometry.

CHAPTER XII

CUBIC INVOLUTIONS AND THE RATIONAL CUBIC CURVE

162. Definition of cubic involution.—The theory of involution as applied to pairs of points (Chap. VII) admits of generalization in several directions.[1] We shall develop the most natural extension in the binary domain. A characteristic property of a quadratic involution is that all pairs of points belonging to it are harmonic to a fixed pair,—the double points of the involution. But when pairs of points are in question harmonic and apolar are equivalent terms which suggests the definition: *A cubic involution consists of all triads of points (elements in the binary domain) apolar to a fixed triad.*

To find an expression for the involution, suppose that the given triads are roots of the equation

$$f\colon a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0 \tag{1}$$

where the a's are fixed constants. And let all sets in the involution be represented by the equation

$$\varphi\colon \alpha_0x^3 + \alpha_1x^2 + \alpha_2x + \alpha_3 = 0 \tag{2}$$

where the α's are undetermined. If φ is apolar to f we must have

$$a_0\alpha_3 - a_1\alpha_2 + a_2\alpha_1 - a_3\alpha_0 = 0. \tag{3}$$

[1] Thus a projective transformation of period two in the plane, in space or in higher dimensions is frequently called an involution. Less frequently a pencil of curves (*e. g.*, a pencil of conics) is called an involution of curves. Likewise a pencil of algebraic forms in any number of variables would be an involution of forms in this sense.

The involution is therefore defined by φ provided the coefficients satisfy equation (3).

A more convenient form can be found as follows. Denoting by s_1, s_2, s_3 symmetric functions of the roots x_1, x_2, x_3 of φ, we have $s_1 = -\alpha_1/\alpha_0$, $s_2 = \alpha_2/\alpha_0$, $s_3 = -\alpha_3/\alpha_0$. Substituting in (3) we obtain the equation

$$a_0 s_3 + a_1 s_2 + a_2 s_1 + a_3 = 0. \tag{4}$$

Hence *all triads of points satisfying* (4) *are apolar to the fixed triad f and constitute therefore a cubic involution.*

Now equation (4) is linear in each of the x's, consequently if two of them are chosen the third is given by a linear equation. Hence *when two points of a triad are specified the third is uniquely determined.* This property of the involution is indicated by the notation $I_{2,1}$.

It follows immediately from the definition or from the theorem just stated that an $I_{2,1}$ contains ∞^2 triads of points.

163. Special sets.—We shall first examine the involution for *multiple points, i. e.*, points formed by the coincidence of two or more elements of a set. Suppose that all three points of a set fall together in a *triple point.* Then $x_1 = x_2 = x_3 = x$ and equation (4) of the preceding section reduces to the form f itself. Hence

1°. *There are precisely three triple points in the involution, namely the roots of f.*

In other words there are three points, represented by the linear factors of f, each of which taken three times constitutes a set in the involution. (*Cf.* 4°, §99.) Moreover

2°. *The points f as a whole form a set in the involution.*

For f is of odd order and therefore self-apolar.

We shall get a *double point* if $x_2 = x_3 = x$. The equation of the involution becomes

$$a_0 x_1 x^2 + a_1(x^2 + 2x_1 x) + a_2(2x + x_1) + a_3 = 0 \tag{1}$$

a quadratic in x. Consequently associated with every x_1 are two double points of the involution, given by the quadratic (1), and there are ∞^1 choices for x_1. Or

3°. *The involution contains a single infinity of double points.*

Finally a cubic has a unique apolar quadratic (§100), *i. e.*,

4°. *There is one pair of points belonging to the involution, namely the Hessian pair of the triple points.*

This simply means that the Hessian points and an *arbitrary* third point form a set in the involution. For this reason the Hessian points are called a *neutral pair* of the involution.

164. We have seen that connected with every cubic involution

$$\alpha s_3 + \beta s_2 + \gamma s_1 + \delta = 0 \qquad (1)$$

is the cubic equation

$$\varphi \colon \ \alpha x^3 + 3\beta x^2 + 3\gamma x + \delta = 0, \qquad (2)$$

obtained by letting the three points of a set coincide. On the other hand the equation (2) determines the cubic involution since (1) is merely the polarized form of φ.

A cubic involution is also determined by three linearly independent triads. Let the triads be represented by

$$f_i \colon \ a_i x^3 + b_i x^2 + c_i x + d_i = 0, \quad i = 1, 2, 3. \quad (3)$$

If these belong to the involution (1) they must be apolar to the triple points φ, *i. e.*,

$$a_i\delta - b_i\gamma + c_i\beta - d_i\alpha = 0. \qquad (4)$$

Now (4) are linearly independent in virtue of the hypothesis that the f's are linearly independent. These three equations therefore just suffice to determine the ratios of the coefficients in (1). Q. E. D.

Eliminating α, β, γ, δ from equations (1) and (4) we have the involution in the determinant form

$$\begin{vmatrix} 1 & -s_1 & s_2 & -s_3 \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} = 0. \qquad (5)$$

Or to write the involution in terms of the three base forms which determine it, we recall (§99, 2°) that φ is apolar to every cubic in the linear system built on the forms f. This suggests a *second definition:*

A cubic involution is represented by a linear system of binary cubics

$$\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 = 0. \qquad (6)$$

Equation (6) contains two essential constants and we see again that two points of a set will determine the remaining one.

165. In particular *a pencil of cubics*

$$f_1 + \lambda f_2 = 0 \qquad (1)$$

is said to form an involution since it is a special case of (6) above. We could define this type as all triads of points apolar to *two* fixed triads. Since the pencil contains a single parameter one point of a set will determine the other two. Accordingly the involution is denoted by $I_{1,2}$.

If f_1 and f_2 are general and wholly independent the only special sets in the involution are four double points given by the Jacobian of the two base forms (§102, 3°). If however the involution contain a cubic and its cubicovariant the only special sets are two triple points given by the Hessian of the cubic (§115).

Again two of the double points may come together to form a triple point while the others remain distinct. Then

taking the triple point for 0 and one of the double points as ∞ the involution can be written in the form

$$x + a + \lambda x^3 = 0. \tag{2}$$

Finally if the two base cubics have a common factor all triads of the involution have a common element and the involution is *singular* or *degenerate.*

EXERCISES

1. An $I_{2,1}$ is determined by (a) two sets and one triple point, (b) two triple points and one set.

2. An $I_{1,2}$ is determined by two triads either or both of which may be a triple point. Or either set may contain a double point.

3. Write the triple points (implicitly) of the involution (5) §164.

4. Write the $I_{2,1}$ determined by the three sets (a) $3x^2 = 0$, $3x = 0$, $x^3 + 1 = 0$, (b) $x_1^3 = 0$, $3x_1^2x_2 + 3x_1x_2^2 = 0$, $x_2^3 = 0$, (c) $t^3 = 0$, $t(1 - at^2) = 0$, $1 - at^2 = 0$. Find the triple points of each.

5. Write the involutions whose triple points are (a) $t^3 - 1 = 0$, (b) $x^3 + x^2 - 2x - 1 = 0$, (c) $\pm i$, 2, (d) 0, 1, ∞.

6. In the $I_{1,2}$ $x + a + \lambda x^3 = 0$, find the other two points of a set when one point is (a) 1, (b) k.

7. If an $I_{2,1}$ is $x^3 + \lambda_1(x - 1)^3 + \lambda_2 = 0$ find the remaining point of the set determined by the pair (a) -1, 2, (b) 2, 3, (c) ω, ω^2. Repeat for the involution $s_1 - 2s_3 = 0$.

8. Find the condition that the four sets of points

$$a_ix^3 + b_ix^2 + c_ix + d_i = 0, \qquad i = 1, 2, 3, 4$$

belong to a cubic involution.

9. The involution of which a, b, c are triple points can be written

$$\lambda_1(x - a)^3 + \lambda_2(x - b)^3 + \lambda_3(x - c)^3 = 0.$$

10. If a, b, c, are the triple points and x, y, z are a variable triad, the involution can be written in the form $(x - a)(y - b)(z - c) + (x - b)(y - c)(z - a) + (x - c)(y - a)(z - b) = 0$. (Expand and compare with (2), §99.)

11. Generalize the results of §§162–4 for four points. Do the 4-fold points form a set in the involution? The $I_{3,1}$ contains ∞^1 neutral triads. What is the condition that it have a neutral pair? How many double points and triple points does the involution contain?

166. Singularities on rational curves.—It is assumed that the student is already familiar with the fundamental singularities of curves.[1] But some supplementary remarks with special reference to rational curves are in order. The elementary singularities of curves are four,—(1) the *double point* or *node*, (2) the *cusp* or *stationary point*, (3) the *double line*, *double tangent* or *bitangent*, (4) the *point of inflexion*, *flex* or *stationary line*.

In considering singularities it is desirable to think of the curve as generated by a lineal element made up of a point x and a line u which are incident (§12). Singularities arise from peculiarities in the behavior of either the describing point or line. It is exceptional for the point x to pass twice through the same position and a double point is formed. Again as x traverses an ordinary region of the curve it moves along the line u in the same direction. At a cusp however the point comes to a dead stop, reverses its sense of motion along the line and recedes from the opposite side of the cusp tangent. But at neither a node nor a cusp is there any peculiarity in the motion of the line u for it neither passes twice through the same position nor does it change its sense of rotation about the point x. If it is rotating clockwise as it approaches a node or a cusp it continues in the same sense while those singularities are being described.

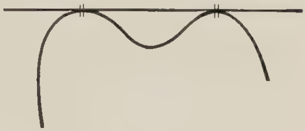
Dually when the tracing line comes after a finite interval to coincide a second time with a given line a double line results which is thus a *line* singularity. Again if u is rotating about x in one sense as it approaches an inflexional tangent, when it reaches the inflexional tangent it comes to a complete stop, reverses its sense of rotation and proceeds with the point of inflexion on its opposite side. But at neither a contact of a double tangent nor at a point of

[1] See for example Granville's Calculus, Chap. XIX.

inflexion is there anything extraordinary in the motion of the point x.

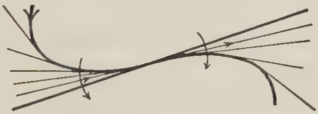
It appears thus that the double point and the cusp are point singularities while their respective duals the double line and the flex line are line singularities.[1] There is also

Double Point



Double Line

Stationary Point (Cusp)



Stationary Line (Flex)

an analytic basis for this classification since a general equation in point coördinates represents a curve which possesses neither a double point nor a cusp but one which has both flexes and double lines.[2] Dually a general equation in line

[1] In the interest of dualistic nomenclature the cusp and flex tangent are sometimes called respectively stationary point and stationary line on account of the momentary pause in the point x or line u as they are being described.

[2] A curve must be of order greater than 2 in order to have a flex and of order greater than three to have a double tangent.

coördinates will represent a curve without double lines or flex lines but one with both cusps and double points.[1]

Any line through a double point cuts the curve in two coincident points there but the two nodal tangents have three coincident intersections at the double point. Likewise any line on a cusp cuts the curve in two coincident points while the cusp tangent meets in three coincident points. An ordinary line on a flex cuts the curve in a single point at the flex but the flex tangent has three coincident intersections there. Four of the intersections of a double tangent are accounted for at the two contacts leaving $n - 4$ additional intersections if the curve is of order n.

Dually a double tangent counts for two common lines of the curve and any point on the double tangent while if the point is at either contact the double tangent counts for three common lines. Thus from an arbitrary point of a double line can be drawn in addition to the double line $m - 2$ tangents to a curve of class m but from a contact of a double line can be drawn but $m - 3$ additional tangents. Similarly from a point on a flex tangent can be drawn $m - 2$ tangents besides the flex tangent while from the flex point can be drawn but $m - 3$ additional tangents. The cusp tangent counts for three tangents from the cusp to the curve but it counts for a single tangent from any other point on the cusp tangent. From a double point apart from the nodal tangents can be drawn only $m - 4$ tangents to the curve.

Coming now to rational curves we assign alike to the point x and the line u the parameter t so that the point and line belong at once to the ternary and the binary domain and we must attend to the double rôle played by each.[2]

[1] A line curve must be of class >2 to have a cusp and of class >3 to have a double point.

[2] Although the point and line have the same parameter, the parametric point and line equations of the curve are as we have seen distinct. This

As the parameter t ranges continuously throughout the number system from $-\infty$ to $+\infty$ the point x moves continuously along the curve returning ultimately to the initial position. The point thus describes the complete curve in a single circuit.[1] Moreover the parameter is spread along the curve in an orderly manner,—exactly as the numbers in a coördinate system along a line. The numbers 0 and ∞ divide the curve into two regions one of which carries all the positive numbers as parameters the other all the negative numbers.

It is evident now that the foregoing exposition concerning x and u as ternary elements will not apply to them as binary elements t. Thus while a double line has a single set of ternary coördinates it has two distinct contacts with distinct parameters. Likewise an ordinary double point has two distinct parameters, one for each nodal tangent. An ordinary tangent cuts the curve in two coincident points x at the contact and also in two coincident points t. A line through a double point however cuts the curve in two coincident points x but not in coincident t's. A nodal tangent cuts out three coincident x's but one t doubly and the other simply.

When a double point becomes a cusp the two parameters coincide. Any line on a cusp cuts the curve in two coincident points t as well as two coincident points x and the cuspidal tangent cuts out three coincident t's as well as three coincident x's. A dual statement holds for a stationary line. Generally coincident t's in the intersections of

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is not surprising for the parametric equations simply express the ternary coördinates in terms of the parameter and the ternary coördinates satisfy different functional relations, *viz.*, the ternary point and line equations of the curve.

[1] On this account rational curves are sometimes called *unicursal*. There is an apparent exception in the case of curves with real asymptotes but each apparent branch unites with some other at infinity to form a connected whole.

curves imply a like number of coincident x's but the converse is not true. Or again coincident t's arise only in connection with consecutive points which need not be true of the x's.

It can be shown that all singularities of curves are compounded from the four fundamental ones. A *triple point* has a single set of ternary coördinates but three parameters which are distinct in general. It is equivalent to (a) three ordinary double points, (b) when two of the parameters coincide, one cusp and two double points, (c) when all three parameters coincide, two cusps and one double point.

The contrast between the ternary and binary theory is exhibited most effectively by means of a parallel summary.

Relation of a Line to a Curve as Indicated by

Ternary Criteria	*Binary Criteria*
When the ternary point equations of a curve and a line are combined,	When the ternary equation of a line is combined with the parametric point equations of a curve,
two coincident x's signify that the (a) line is tangent (b) line is on an (ordinary) double point (c) line is on a cusp.	*two coincident* t's imply that the (a) line is tangent (at an ordinary point or to a branch of a multiple point) (b) line is on a cusp (which may lie at a multiple point).
three coincident x's mean that the (a) line is flex tangent (b) line is cusp tangent (c) line is nodal tangent (d) line is on a triple point (any variety).	*three coincident* t's mean that the (a) line is flex tangent (b) line is cusp tangent (c) line is on a triple point with three coincident parameters.

two pairs of coincident x's imply that the
(*a*) line is a bi-tangent
(*b*) tangent line is on a node
(*c*) tangent line is on a cusp
(*d*) line is on two nodes
(*e*) line is on two cusps
(*f*) line is on a node and a cusp.

two pairs of coincident t's imply that the
(*a*) line is a bi-tangent
(*b*) tangent line is on a cusp
(*c*) line joins two cusps (either distinct or which may fall together at a multiple point).

Dual Criteria

When the ternary line equations of a curve and a point are combined, *two coincident* u's imply that the
(*a*) point is on the curve
(*b*) point is on a double line
(*c*) point is on a flex line.

three coincident u's mean that the
(*a*) point is a cusp
(*b*) point is a flex
(*c*) point is a contact of a double tangent
(*d*) point is on a triple line (either of three varieties).

two pairs of coincident u's signify that the
(*a*) point is a double point

When the ternary equation of a point is combined with the parametric line equations of a curve, *two coincident* t's imply that the
(*a*) point is on the curve
(*b*) point is on a flex tangent.

three coincident t's mean that the
(*a*) point is a cusp
(*b*) point is a flex
(*c*) point is on an undulation tangent.

two pairs of coincident t's signify that the
(*a*) point is an ordinary node

(*b*) point is on the curve where a double line cuts (not at the contact)

(*c*) point is on the curve where a flex tangent cuts again

(*d*) point is intersection of two double lines

(*e*) point is intersection of two flex tangents

(*f*) point is intersection of a double line and a flex line.

(*b*) point is on the curve where a flex tangent cuts again

(*c*) point is intersection of two flex tangents (which may be distinct or which may coincide to form part of a multiple line.)

EXERCISES

1. What is the dual of a *flecnode*, *i. e.*, a double point formed by the tracing point passing through a point of inflexion? This singular line cuts the curve in five points at its two contacts. Dually the flecnode tangents count for five tangents, *i. e.*, from a flecnode can be drawn only $m - 5$ additional tangents.

2. What is the dual of a *biflecnode*,—a double point both of whose tangents are flex tangents? (This is the type of double point on the lemniscate.) How many tangents, in addition to the nodal tangents, can be drawn to a curve from a biflecnode?

3. When a simple branch of a curve passes through a cusp a triple point is formed. What is the variety? Dualize.

4. Dualize the other two types of triple points. How many intersections are accounted for at the contacts of each singular line corresponding to the three varieties of triple points? How many tangents, in addition to the tangents at the triple point, can be drawn to a curve from each variety of triple point? Draw each type of triple point and the dual singularity.

5. Verify the ternary criteria for three coincident x's by considering (*a*) the cubical parabola, (*b*) the semicubical parabola, (*c*) the folium of Descartes, (*d*) curve (*a*) Ex. 8, §158.

6. What is the dual of a *double-flex* tangent,—*i. e.*, a line which is tangent at each of two distinct points of inflexion? How many

tangents can be drawn to a curve from the point singularity in question?

7. How many tangents can be drawn to a curve from an ordinary point on a double-flex tangent, from one of the contacts?

8. Two curves have a common double point. How many intersections are accounted for? How many if the curves have a common node and one common nodal tangent, (*b*) two common nodal tangents? Dualize.

9. A conic is inscribed in (*a*) five flex tangents, (*b*) five double tangents, (*c*) five nodal tangents, of a curve. How many common lines are accounted for?

10. A curve has three flex tangents which touch again with a simple contact. A conic touches the curve at the three simple contacts of the singular lines and in two other points. How many common lines are accounted for?

11. A conic is inscribed in five double-flex tangents of a curve. How many common lines are accounted for?

12. Curve (*b*) Ex. 7, §160 has a biflecnode at the vertex u_3. For every line on u_3 cuts out the parameters 0 and ∞ while the axis x_1 cuts out ∞ three times and the axis x_2 cuts out 0 three times. Examine the nature of the singularity at u_3 of curves (*c*), (*d*), (*e*) of the same exercise.

13. A rational quartic curve has a triple point consisting of a cusp with a branch through it. Assigning to this point the parameters 0 and ∞ and taking the two tangents as two axes of reference show that the equations of the curve can be written $x_1 = t^3$, $x_2 = t^2$, $x_3 = t^4 + at + 1$. (The term in t^3 and that in t^2 in x_3 can be removed and the constant term made 1 by linear transformations on the x's and on t.)

14. If a rational quartic has three cusps with parameters 0, 1, ∞, write its parametric point equations referred to the cusp triangle.

167. Applications to the rational cubic.—A beautiful illustration of the theory of cubic involutions if furnished by the rational cubic curve in the plane. Such a curve is defined parametrically by the three equations

$$\begin{aligned} x_1 &= f_1 \equiv a_1t^3 + b_1t^2 + c_1t + d_1 \\ x_2 &= f_2 \equiv a_2t^3 + b_2t^2 + c_2t + d_2 \qquad (1) \\ x_3 &= f_3 \equiv a_3t^3 + b_3t^2 + c_3t + d_3 \end{aligned}$$

where the f's are linearly independent forms.

Any line $(ux) \equiv u_1x_1 + u_2x_2 + u_3x_3 = 0$ cuts the curve in three points given by

$$u_1f_1 + u_2f_2 + u_3f_3 = 0 \tag{2}$$

or

$$(au)\,t^3 + (bu)t^2 + (cu)t + (du) = 0.$$

But when u is a parameter (2) is a linear system of cubics which comprises all line sections, *i. e.*,

The ∞^2 line sections of a rational cubic belong to a cubic involution $I_{2,1}$.[1]

Accordingly the properties of the involution can be translated at once into properties of the curve. First of all there are three triple points in the involution or three cubics in (2) which are cubes. That is to say there are three lines each of which cuts the curve in coincident points with coincident parameters. Hence

The rational cubic has three points of inflexion whose parameters are the triple points of the involution of line sections.

Since the condition that a triad of points belong to the involution is simply that it be apolar to the triple points we may say that the condition that three points of the curve lie on a line is that their parameters be apolar to the flex parameters. Or more briefly

Three points are on a line if and only if they are apolar to the flexes.

Now the triple points (flex parameters) are themselves a set of the involution, *i. e.*,

The flexes of the rational cubic are collinear.

Again there is one pair of points in the involution,—the neutral pair. In the language of line sections this says

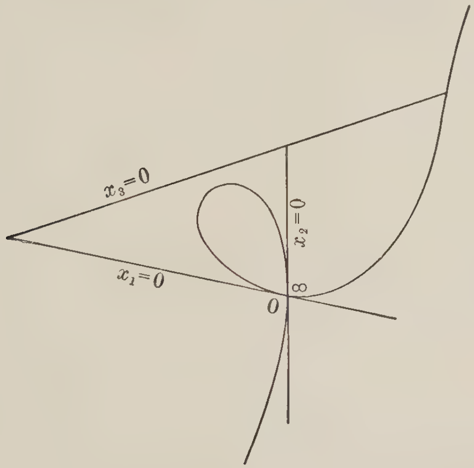
[1] The student must not forget that points as elements in the involution are here regarded as belonging to the binary domain,—that in short properties of the involution are properties of points t (see §129).

that there is one pair of t's collinear with any third t. This can happen only when the parameters in the neutral pair are attached to the same point which is thus a double point or *node* of the curve. Therefore

The rational cubic has a double point whose parameters, the neutral pair in the involution, are given by the Hessian of the binary cubic naming the flexes.

Any tangent to the curve of course cuts out a triad of the involution with a double point. We saw (§163, 3°) that every t is an element in two triads which have double points. This means that two tangents can be drawn to the curve from an arbitrary point t on the curve.[1]

168. The canonical form of the rational cubic.—The properties of the curve already developed lead us to a



canonical form of the equation which will facilitate our further study. As in the case of the line and the conic we may assign parameters at random to any three points of

[1] These are exclusive of the tangent at t which counts for two, the rational cubic being of class 4.

the curve. Accordingly we shall take -1, $-\omega$, $-\omega^2$ as the flex parameters and choose the line of flexes as one side of the triangle of reference. Then the parameters of the node which are the Hessian of the cubic $(t^3 + 1)$ representing the flexes will be 0 and ∞. Thus a natural choice for the other sides of the reference triangle is the nodal tangents, one of which cuts out the parameter 0 twice and ∞ once while the other cuts out 0 once and ∞ twice. The equation of the curve may then be written in the canonical form[1]

$$\begin{aligned} x_1 &= 3t^2 \\ x_2 &= 3t \\ x_3 &= t^3 + 1. \end{aligned} \tag{1}$$

The triple points of the involution of line sections are given by the flex cubic $t^3 + 1 = 0$. Polarizing this *the condition that three points* t_1, t_2, t_3 *belong to the involution, i. e., lie on a line* is

$$s_3 + 1 \equiv t_1t_2t_3 + 1 = 0. \tag{2}$$

In fact any line $(ux) = 0$ meets the curve in the three points

$$u_3t^3 + 3u_1t^2 + 3u_2t + u_3 = 0 \tag{3}$$

whence obviously

$$s_3 = -1.$$

If two of the points in (2) coincide, say $t_1 = t_2 = \tau$, the equation becomes if we write t for t_3

$$\tau^2t + 1 = 0. \tag{4}$$

The line u is now tangent at τ while t is the remaining intersection of the tangent, called the *tangential* of τ. (4) is thus the relation connecting a point τ and its tangential t. It is easy to prove hence that

The tangentials of three points of a line lie on a second line,—called the *satellite* line of the first.

[1] This form has the advantage of simplicity but the disadvantage of being unsymmetrical. We could get a symmetrical form by taking the flex tangents as reference lines.

For if the parameters of two points are t_1, t_2 the point collinear with them is (by (2)) $-1/t_1t_2$. The tangentials of the three (by (4)) are $-1/t_1^2$, $-1/t_2^2$, $-t_1^2t_2^2$ the product of which is -1. Hence the three points are on a line.

Again the contacts τ of tangents from t are given by the quadratic $\tau^2 + 1/t = 0$, a pair of points evidently harmonic with 0 and ∞. We have thus a simple proof of the elegant theorem

The parameters of the pair of contacts of tangents from an arbitrary point of the rational cubic belong to a quadratic involution whose double points are the nodal parameters.

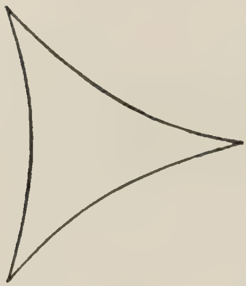
To establish the theorem synthetically consider the pencil of lines on any point t of the curve. Every line of the pencil cuts the curve in a pair of points, exclusive of t, which are represented by a binary quadratic. And the pencil of lines will cut out ∞^1 pairs of points given by a pencil of quadratics. In other words the pairs of points constitute a quadratic involution whose double points are manifestly the contacts of tangents from t. But there is one line of the pencil passing through the node, *i. e.*, the nodal parameters are a pair in the involution and therefore harmonic with its double points. Or we may say that the double points of the quadratic involutions associated with points t of the curve are themselves pairs in an involution the double points of which are the nodal parameters.

The rational class cubic. Dually the rational curve of class three is a rational point curve of order four having one double line and three cusps—whose tangents meet at a point. The parameters of the contacts of the double line are the roots of the Hessian of the binary cubic naming the cusps. If we assign to the cusps the parameters 1, ω, ω^2, the contacts of the double line are 0 and ∞. Then taking for reference 3-point the contacts of the double line and the

point of intersection of the cusp tangents the equations of the curve assume the canonical form

$$u_1 = t^2,\ u_2 = t,\ u_3 = t^3 - 1. \tag{5}$$

It will be interesting to identify two familiar metrical forms of the rational class cubic. The equation in absolute coördinates $x = 2t + 1/t^2$ (Ex. 24, (l) §156) represents a hypocycloid of three cusps, sometimes called Steiner's hypocycloid and by Prof. Morley a *deltoid*. The curve is of



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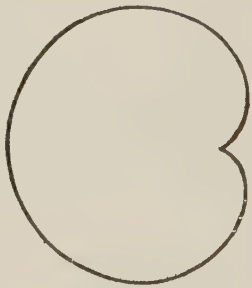

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order four but of class three. For the homogeneous line equations, on removing the factor $t^3 - 1$ corresponding to the cusps, reduce to the canonical form (5), the signs of the u's being immaterial.

Again the curve $x = 2t - t^2$ (Ex. 24, (d), §156) represents a cardioid. The homogenous line equations, after removing the common factor are

$$u_1 = 1,\ u_2 = -t^3,\ u_3 = 3(t^2 - t) \tag{6}$$

hence the cardioid is a rational class cubic. Now equations (6) are just the form we should be led to if we assigned to the cusps the parameters 0, 1, and ∞ and adopted for reference 3-point two cusps and the point of intersection of cusp tangents. In other words the curve is a *general* rational

class cubic. Since the two line curves here considered differ only in the way the reference elements, binary and ternary, are chosen we may say

The cardioid and the deltoid are projectively equivalent curves.

The great dissimilarity in the metrical aspect of the curves is due as usual to the difference in imaginary and infinite elements. Thus the cusps of the deltoid are real but the double line (£) is *isolated*, with contacts at the circular points. The cardioid on the other hand has a real double line and one real cusp while the other cusps are imaginary, lying at I and J.

The student will find it an instructive exercise to derive one curve from the other by linear transformations of the parameter and the ternary coördinates. (It will be simpler to work with the line equations.)

EXERCISES

1. Write in determinant form the binary cubic that gives the flex parameters of the general rational cubic (1) §167. How would you find the nodal parameters?

2. Write the condition that the points given by the cubic $\alpha t^3 + \beta t^2 + \gamma t + \delta = 0$ be on a line for the curve in Ex. 1.

The remaining exercises apply to the curve in canonical form, (1) §168 unless otherwise stated. Much of the theory of Chap. X, Part I is available.

3. Find the ternary point equation of the curve. (Eliminate t by inspection, §124, or by observing that $t = x_1/x_2$.)

4. Find the map equation and the parametric line equations of the curve.

5. Find the ternary line equation by writing the discriminant of (3), §168, as a cubic in t.

6. Find the equation of the line joining the points t_1, t_2.

7. Find the equations of the flex tangents. Show that the triangle of flex lines is fully perspective with the reference triangle. (See §61, Ex. 10.)

8. Show directly that the points of inflexion (parameters -1, $-\omega$, $-\omega^2$) are on a line.

9. Write the parametric point equations of the curve referred to the flex tangents (taking the flex parameters as before). Find the ternary point equation of this curve. (Transform the triangle of reference (§62) to the lines in Ex. 7.)

10. Write the parametric equations of the curve referred to the flex tangents when the parameters of the flexes are 1, 0, ∞.

11. Assign to the flexes the parameters 0, ∞, 1. Then taking for reference triangle the tangents 0 and ∞ and the line of flexes write the parametric equations of the curve. Find the ternary point equation.

12. Repeat Exs. 4–6 for the curve written as in Exs. 9, 10, 11. Also when the equations of the curve are $x_1 = t^3$, $x_2 = t(1 - at^2)$, $x_3 = 1 - at^2$. Find the ternary point equation of this curve.

13. Find the condition that three points be on a line for the curves in Ex. 12. Also find the nodal parameters. What are the relations connecting the parameter of a point and its tangential?

14. Write the binary quartic giving the parameters of the 4 tangents that can be drawn from an arbitrary point x of the plane. Calculate the invariants I_2 and I_3 of this quartic in t. What is the geometric interpretation of the curves $I_2 = 0$ and $I_3 = 0$? (§117.)

15. Write the discriminant (§116) of the quartic in t of Ex. 14. Show that this sextic in x represents the point equation of the curve multiplied by an extraneous factor,—the product of the three flex tangents.

16. The first polar (§140, Ex. 8) of a point y with respect to the cubic, Ex. 3, is a conic which passes through the node and cuts out the contacts of the four tangents from y.

17. The polar of a point of inflexion breaks up into the flex tangent and the line joining the node to the contact of the tangent from the flex. (This second line is a *harmonic polar.*)

18. Find the quadratic involution (on the parameter) cut out by the pencil of lines on the point t_1. Find the quadratic involution for the curves in Ex. 12. (Use the results of Ex. 13.)

19. Find the equation of the satellite line of the line (u_1, u_2, u_3). *Ans.* $(3u_2^2 - 2u_3u_1)x_1 + (3u_1^2 - 2u_2u_3)x_2 + u_3^2x_3 = 0$. (Winger, *On the satellite line of the cubic*, Amer. Jour. Math., 1920.)

20. If a line touches the curve, its satellite touches at the tangential point. What is the satellite of (a) a flex tangent, (b) the line of flexes, (c) a nodal tangent? Prove analytically and synthetically.

21. For variable u and constant x the equation in Ex. 19 represents a conic,—the locus of lines whose satellites are on x.

22. The discriminant of the conic in Ex 19 is to a factor the point equation of the cubic. Interpret.

23. The three pairs of contacts of tangents from three collinear points of the curve are the complete intersections of four lines,—each of which has the original line for satellite.

24. Any cubic curve with a double point is rational. For if $u = 0$ and $v = 0$ are two lines meeting at the double point every line of the pencil $u + tv = 0$ will cut the curve in a third point. Thus through t we have a (1, 1) correspondence between the lines of a pencil and the points of the curve and the coördinates of the points of the curve are expressible as rational functions of t.

25. Find parametric equations of the following cubic curves: (a) the cissoid $y^2(2a - x) = x^3$, (b) the strophoid $(a - x)y^2 = (a + x)x^2$, (c) $(x_1^2 - x_2^2)x_3 = x_1^3$, ($d$) $x_3x_2^2 = x_1(x_1 - ax_3)^2$ (node at $(a, 0, 1)$).

26. Find the Hessian (§101) of the rational cubic Ex. 3. The Hessian is a rational cubic having the same double point, with the same tangents, and the same flexes as the original.

27. Any curve of order $n + 1$ with an n-fold point is rational. (This is a sufficient but not a necessary condition.) Dualize.

28. Dualize the satellite property. Dualize Exs. 9, 10, 14, 16, 17, 20, 21, 23, 24, 26, 27.

169. Geometry on the rational cubic.—Consider a pencil of lines on the double point of the curve. Every line of the pencil cuts the curve in one point in addition to the two intersections at the node. If then we correlate each line of the pencil with its third point of intersection we shall have a (1, 1) correspondence between lines of the pencil and points of the curve. If we write the pencil

$$x_1 = \lambda x_2 \tag{1}$$

then the parameters of the point in which the line λ meets the curve are, substituting the values of the x's from the canonical form in (1)

$$3t(t - \lambda) = 0, \text{ or } t = 0, \infty, \lambda.$$

Thus in addition to nodal intersections the line λ of the pencil cuts the cubic in the point $t = \lambda$. In other words *the parameter (on the curve) of a point is identical with the*

parameter (in the pencil) of its correlative line. It follows that to construct a point t of the curve it will suffice to draw the line t of the pencil.

To illustrate, the binary cubic f will represent three points t_1, t_2, t_3 on the curve which are cut out by the lines t_1, t_2, t_3 of the pencil on the node. The harmonic conjugate of the line t_i with respect to the pair t_j, t_k is a line t_i' cutting the curve in a cubicovariant point. The lines t_i, t_i' $(i = 1, 2, 3)$ are pairs in a quadratic involution whose double lines cut the curve in the Hessian points of f. (See §115.)

Suppose now that the binary cubic f is the flex form $t^3 + 1$. Then we saw that the Hessian represents the nodal parameters. Now the contacts of tangents from the flex -1 are (§168, (4)) ± 1, one of which is the flex itself. Hence from each point of inflexion can be drawn in addition to the flex tangent itself but one tangent to the curve. And *the contacts* 1, ω, ω^2 *of the three tangents from the flexes are given by the cubicovariant* $t^3 - 1$. Or to interpret the system when f represents lines of the pencil on the node we may say:

The pair of lines joining the double point of a rational cubic to any point of inflexion and to the point of contact of the tangent from the flex are conjugate lines in a quadratic involution whose double lines are the nodal tangents.

170. As an example of an $I_{1,2}$ may be mentioned the involution set up on the rational cubic by the pencil of lines $(ux) + \lambda(vx) = 0$. The lines will cut out a pencil, *i. e.*, an $I_{1,2}$ of binary cubics, say $u + \lambda v = 0$. The involution in general will have four double points, namely the contacts of tangents from the center of the pencil, given by the Jacobian of u and v.

The involution may be special for special positions of the point which determines it. Thus if the point is on the curve the involution is singular—degenerates to a quadratic involution. We ask now whether there are any line sec-

tions u whose cubicovariant points u' are also line sections. If so these lines may be taken as base lines of the special involution $u + \lambda u' = 0$. A characteristic of such an involution is that the only special sets are two triple points (§115). It follows that the center of the pencil must be the intersection of two inflexional tangents. Consequently

The locus of lines u which cut out binary cubics whose cubicovariant points lie on a line u' consists of the vertices of the triangle of flex tangents. Moreover (§115) *lines u and u' in any pencil belong to a quadratic involution of lines whose double lines are the two flex tangents of the pencil.*

171. The general involution of order n in the binary domain we define as all sets of n points apolar to a fixed set of n. If the fixed set is represented by the equation

$$f \equiv a_0x^n + na_1x^{n-1} + \binom{n}{2}a_2x^{n-2} + \cdot \cdot \cdot + \binom{n}{n-1}a_{n-1}x + a_n = 0 \quad (1)$$

then as in §162 the involution can be written

$$a_0s_n + a_1s_{n-1} + a_2s_{n-2} + \cdot \cdot \cdot + a_{n-1}s_1 + a_n = 0, \quad (2)$$

where the s's refer to the elementary symmetric functions of the coördinates $x_1, x_2, \ldots x_n$ of the n points in a set. The involution (2) is denoted by $I_{n-1,1}$ since evidently $n - 1$ elements of a set determine the remaining one.

If the x's all coincide (2) reduces to (1), hence *an $I_{n-1,1}$ contains n n-fold points which constitute the fixed set to which all sets of the involution are apolar.*

Thus associated with every equation (1) is an involution (2), the polarized form of the equation. The involution is really the more comprehensive idea, the equation simply defining the coincidence points of the involution.

Now there are n linearly independent forms f_i of order

n apolar to a given form f of order n (§99, 3°). Hence an $I_{n-1,1}$ can be defined by an $(n - 1)$-parameter family of binary n-ics

$$f_0 + k_1f_1 + k_2f_2 + \ . \ . \ . + k_{n-1}f_{n-1} = 0 \qquad (3)$$

where f_i are forms apolar to f. That is (3) *represents the involution which is determined by the n linearly independent sets f_i.*

A special involution of order n may be defined as all sets of n points $x_1, x_2, \ . \ . \ . \ x_n$ which are apolar to r fixed sets $f, f', \ . \ . \ . \ f^r$. The x's must then satisfy r equations like (2) so that (3) reduces to the $(n - r)$-parameter family

$$f_0 + k_1f_1 + k_2f_2 + \ . \ . \ . + k_{n-r}f_{n-r} = 0. \qquad (4)$$

Since now $n - r$ points of a set suffice to determine the remaining r the involution is denoted by $I_{n-r,r}$. While the involution is special for $r > 1$ it is identical with (2) when $r = 1$.

172. Some contact conics of the rational cubic.—To find the intersections of the rational cubic with the general conic we substitute the x's from the parametric equations of the cubic ((1) §168) in the ternary equation of the conic. The result is a sextic in t in which obviously the coefficient of t^6 is the same as the constant term, *i. e.*, the product s_6 of the roots of the sextic is unity. Conversely if the parameters $t_1, \ . \ . \ . \ t_6$ of six points satisfy the relation $s_6 = 1$ the six points lie on a conic. For a conic can be passed through five of the points $t_1, \ . \ . \ . \ t_5$ which will cut the curve in a sixth point t' such that $t_1t_2t_3t_4t_5t_6' = 1$, hence $t_6' \equiv t_6$. Thus

The necessary and sufficient condition that six points of the rational cubic lie on a conic is that their parameters satisfy the relation

$$s_6 = 1 \qquad (1)$$

or that they belong to the sextic involution $I_{5,1}$ defined by (1).

When two or more t's of a set come together a multiple point of the involution is formed and the corresponding conic has contact[1] with the curve. Passing to the extreme case at once *the 6-fold points of the involution are*

$$t^6 - 1 \equiv (t^3 + 1)(t^3 - 1) = 0 \tag{2}$$

three of which are the flexes while the other three ($t^3 - 1 = 0$) are the contacts of tangents from the flexes. From (2) we infer that there are six points at which conics can be drawn having 6-point contact. But the flex tangents taken twice will count among these conics, *i. e., there are three points at each of which a proper conic can be drawn to have 6-point intersection.* The conics are called *sextactic conics* and the points *sextactic points.* Therefore

The rational cubic has three sextactic points which are given by the cubicovariant of the binary cubic naming the flexes.

Next suppose that the six points of a set in the involution (1) coincide to form three double points t_1, t_2, t_3. The involution becomes

$$t_1^2 t_2^2 t_3^2 = 1 \quad \text{or} \quad s_3 = \pm 1 \tag{3}$$

and the corresponding conics are *tri-tangent.* But if $s_3 = -1$ the conic is any line repeated since the three t's are then collinear. It is clear therefore that

The rational cubic has ∞^2 *tri-tangent conics whose contacts belong to the involution* $s_3 - 1 = 0$ *whose triple points are the sextactic points.*

In particular since the triple points are themselves a set in the involution, *there is one conic touching at the sextactic points.* This is an important conic which we shall meet again. Among the tritangent conics must be counted the line pairs consisting of two tangents from a point of the

[1] Contact here simply means coincident intersections and will include improper contact as well as ordinary tangency, *i. e.*, coincidence of consecutive points.

curve, hence the point t and the contacts of tangents from t are a set apolar to the sextactic parameters.

EXERCISES

The cubic is to be taken in canonical form unless otherwise stated.

1. Write the cubic involutions $(I_{1,2})$ determined by the vertices u_1 and u_2 of the reference triangle. What are the double points?

2. Write the three involutions determined by the vertices of the flex 3-line and verify that each is of the form $u + \lambda u' = 0$ where u and u' are a binary cubic and its cubicovariant.

3. Write the $I_{1,2}$ determined by a flex tangent and a nodal tangent. What are the special sets in this case?

4. The equation of the line t_1t_2 (Ex. 6, §168) is quadratic in each parameter. Show that the discriminant of this as a quadratic in t_2 is the ternary equation of the pair of tangents from t_1.

5. At each point τ of the rational cubic one conic can be drawn with 5-point contact. (Call the conic a *quintactic* conic and the point a quintactic point.) The conic will cut the cubic simply at a point t. But there are five quintactic conics which pass through t simply. The quintactic points of these five conics lie on a conic with t. (In the fundamental equation (1) §172 let five t's coincide at τ and denote the other by t. The equation becomes $\tau^5 t = 1$, from which the theorem can be read.)

6. Two proper conics can be drawn tangent to the rational cubic at a given point t and having 4-point contact at another point τ. (A tangent from t counted twice will be a degenerate conic of this sort.) The two *quartactic* points τ are collinear with t.

7. From any point $(-1/t^2)$ draw the two tangents t and $-t$ and from each of these points the pair of tangents. The two pairs of contacts of tangents from t and $-t$ are harmonic and form thus with the nodal tangents three mutually harmonic pairs. At either point t or $-t$ two conics touch which have quartactic points at contacts of tangents from the other.

8. Consider the six points t, ωt, $\omega^2 t$ and $1/t$, ω/t, ω^2/t. At each point in either triple three conics *osculate* (have 3-point contact) each of which osculates again at one point in the other triple. We have thus nine conics. The six points lie on a tenth conic. Find its equation.

9. The tangentials of the six points in which a conic cuts the rational cubic lie on a conic.

10. Any cubic curve, general, rational or degenerate, cuts the rational cubic in nine points whose parameters belong to the $I_{8,1} : s_9 = -1$, where s refers to the product of the t's.

11. The condition that $3n$ points of the rational cubic be the complete intersections of a curve C_n of order n is that they belong to the involution $I_{3n-1,1} : s_{3n} = (-1)^n$, where s refers to the product of the parameters.

12. The tangentials of the $3n$ points in which a C_n cuts the rational cubic lie on a C_n. The contacts of tangents to a rational cubic from the $3n$ intersections with a C_n lie on a C_{2n} when n is even. (For an application of these theorems and extensions to other curves see a paper by the author, *Some generalizations of the satellite theory*, Bull. Amer. Math. Soc., Nov., 1919.)

13. Conics with quintactic points at three collinear points of a rational cubic cut the cubic again at three points of a line. Conics with quintactic points at the intersections of a given conic with the rational cubic cut again at six points of a conic, and so on. Generalize when the quintactic conics are replaced by cubic curves with 8-point contact.

14. Extend the results of §167 to the rational quartic curve in space, making use of Ex. 11, §165. The parametric equations of the curve are

$$x_i = f_i \equiv a_i t^4 + 4b_i t^3 + 6c_i t^2 + 4d_i t + e_i, \qquad i = 1, 2, 3, 4$$

where f_i are linearly independent forms. The ∞^3 plane sections $u_1 f_1 + u_2 f_2 + u_3 f_3 + u_4 f_4 = 0$, belong to an $I_{3,1}$ which contains four 4-fold points. Let these be given by $f \equiv at^4 + bt^3 + ct^2 + dt + e$. The condition that four points be on a plane is that they be apolar to f.

There are four points (hyperosculation points), given by $f = 0$ at each of which a plane with 4-point contact can be drawn.

At each point of the curve an osculating plane can be drawn (which accounts for the ∞^1 triple points of the involution). (A space curve can be regarded as a locus of points, a locus of its tangent lines or a locus of its osculating planes.)

The curve has ∞^1 trisecant lines, for the involution contains ∞^1 neutral triads.

The condition that the hyperosculation points lie on a plane is that f be self-apolar, *i. e.*, $I_2 = 0$.

If $I_3 = 0$ the involution has a neutral pair and the curve has a double point.

15. Let f, Ex. 14, be $t^4 + 1$ and find a system of four linearly independent binary quartics apolar to f. (See §99 under 3°.) Hence show that the rational space quartic curve with a double point can be written in the canonical form $x_1 = t^3$, $x_2 = t^2$, $x_3 = t$, $x_4 = t^4 - 1$. What are the parameters of the node? What are the reference planes? The condition that four points be on a plane is $s_4 \, (\equiv t_1t_2t_3t_4) + 1 = 0$.

16. If $f \equiv t^4 + 6kt^2 + 1$ (Ex. 14) the equations of the rational quartic curve can be written in the form $x_1 = t^3$, $x_2 = t^2 - k$, $x_3 = t$, $x_4 = t^4 - 1$, which reduces to the curve in Ex. 15 when $k = 0$. Find the condition that four points lie on a plane.

17. From any point t of the curve in Ex. 15 can be drawn three osculating planes and their points of contact are on a plane with t.

173. The perspective conics of the rational cubic.— Two rational curves S and Σ, the first in points and the second in lines are said to be *perspective* if the point t of S lies on the line t of Σ. If the parametric equations of the two curves are

$$S\colon\ x_1 = f_1(t), \qquad x_2 = f_2(t), \qquad x_3 = f_3(t)$$
$$\Sigma\colon\ u_1 = \varphi_1(\tau), \qquad u_2 = \varphi_2(\tau), \qquad u_3 = \varphi_3(\tau)$$

then the necessary and sufficient condition that they be perspective is that like named point and line be incident, *i. e.*,

$$(ux) \equiv \varphi_1 f_1 + \varphi_2 f_2 + \varphi_3 f_3 \equiv 0, \quad \text{when } \tau = t. \tag{1}$$

We have already seen that *the double point of the cubic is a perspective point* (§169) for lines t of the pencil on the node cut the curve again in points t. We shall now show that

Lines joining pairs of points on the rational cubic whose parameters are in a quadratic involution envelop a conic perspective to the cubic.

Let

$$at^2 + 2bt + c = 0 \tag{2}$$

where the coefficients are fixed constants name the double points of the quadratic involution on the cubic. And let all pairs t_1, t_2 in the involution be given by

$$\alpha t^2 + \beta t + \gamma = 0 \qquad (3)$$

where the coefficients are parameters.

Now the line joining the points t_1, t_2 is by the method of §126

$$(s_1 s_2 - 1)x_1 + (s_1 - s_2^2)x_2 - 3s_2 x_3 = 0 \qquad (4)$$

where $s_1 = t_1 + t_2$, $s_2 = t_1 t_2$. But from (3) $s_1 = -\beta/\alpha$ $s_2 = \gamma/\alpha$. Making these substitutions in (4) we get the line $t_1 t_2$ in the form

$$(\beta\gamma + \alpha^2)x_1 + (\alpha\beta + \gamma^2)x_2 + 3\gamma\alpha x_3 = 0. \qquad (5)$$

This line cuts the cubic in a third point t given by

$$\gamma t + \alpha = 0 \qquad (6)$$

since we must have ((2), §168)

$$t_1 t_2 t = -1.$$

By hypothesis (2) and (3) are apolar, *i. e.*,

$$a\gamma - b\beta + c\alpha = 0. \qquad (7)$$

Thence substituting for β in (5) we obtain

$$(b\alpha^2 + c\gamma\alpha + a\gamma^2)x_1 + (c\alpha^2 + a\gamma\alpha + b\gamma^2)x_2 + 3b\gamma\alpha x_3 = 0 \qquad (8)$$

as the equation of the line cutting out pairs in the involution.

Considering α/γ as a parameter (8) is the equation of a line of a conic since it is quadratic in the parameter. To prove that the conic is perspective it is only necessary to show that the parameter can be so transformed that like named point and line of cubic and conic are incident. This merely requires (by (6)) $\alpha/\gamma = -t$.

Thus we have not only proved the theorem but we have incidentally found the equation of the conic. Replacing α/γ in the map equation (8) by the value of the ratio from

(6) we have as *the parametric equations of the perspective conic in lines*

$$u_1 = bt^2 - ct + a$$
$$u_2 = ct^2 - at + b \tag{9}$$
$$u_3 = -3bt.$$

The conic is determined when the double points of the involution are known. Since the double points are given by a general quadratic (2) we have the theorem of Stahl:[1]

The rational cubic has ∞^2 *perspective conics which belong to a linear system* (8).

By assigning arbitrary values to a, b, c in (9) the equations of all perspective conics are obtained.

The six points in which conics (9) cut the cubic are found by combining the ternary equation of the conics with the parametric equations of the cubic. The binary sextic giving the common points is the square of the cubic

$$bt^3 + ct^2 - at - b = 0. \tag{10}$$

It follows that the perspective conics have each three contacts with the cubic curve which satisfy the relation $s_3 - 1 = 0$, *i. e.*, belong to the $I_{2,1}$ whose triple points are the sextactic points. Since this involution is the same as that to which the contacts of tri-tangent conics belong we conclude that the perspective conics are identical with the system of tri-tangent conics.

We noticed above that there is one conic touching at the sextactic points. Its equation can be found at once from the consideration that (10) must reduce to $t^3 - 1 = 0$. The conditions are $b = 1$, $a = c = 0$, *i. e.*, that the double points of the quadratic involution shall be the nodal parameters.

[1] Zur Erzeugung der ebenen rationalen Curven, *Mathematische Annalen*, Vol. 38 (1891). See also Coble, *American Journal of Mathematics*, Vol. 32, p. 350.

Whence the equations of the conic, which may be called the *involution conic of the node* and which we shall denote by N, are

$$\begin{aligned} u_1 = t^2,\ u_2 = 1,\ u_3 = -3t, \text{ or } u_3{}^2 - 9u_1u_2 = 0 \\ x_1 = 3,\ x_2 = 3t^2,\ x_3 = 2t, \text{ or } 9x_3{}^2 - 4x_1x_2 = 0. \end{aligned} \tag{11}$$

Pairs in the involution are now contacts of tangents from the points of the curve (§168). Moreover it appears from (11) that the conic touches two sides of the reference triangle at the vertices. Therefore

Lines joining pairs of contacts of tangents from the points of the rational cubic envelop a conic N which touches the nodal tangents where they meet the line of flexes and touches the cubic at the sextactic points.

EXERCISES

1. Obtain the parametric line equations of the perspective conics (9) by asking that $\varphi_1 f_1 + \varphi_2 f_2 + \varphi_3 f_3 \equiv 0$, where f_i are the binary cubics in the canonical equations of the rational cubic (1), §168 and where $\varphi_i = a_i t^2 + b_i t + c_i$, $i = 1, 2, 3$.

2. Obtain the map equation and thence the parametric equations of the perspective conics by writing the condition that the quadratic (2) be apolar to the form (4) considered as a quadratic in t_2. (t_1 is then to be the variable parameter.)

3. Write the ternary point equation of the perspective conic (9). Find the intersections with the rational cubic and verify that the conic has three contacts given by (10).

4. The Jacobian of the coefficients in the equation of the satellite line Ex. 19, §168 is equal to u_3N.

5. The three contacts of a perspective conic are given by the Jacobian of the quadratic (2) and the binary cubic naming the flexes. (Ashcraft, *Quadratic involutions on the plane rational quartic*, Waterville, Maine.)

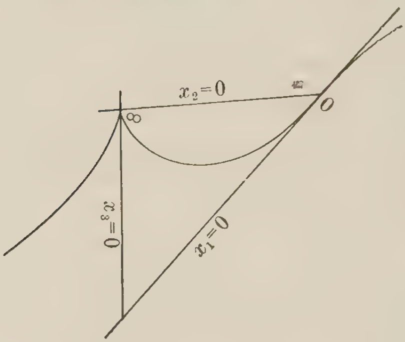
6. Find the perspective conics of the rational cubic (a) $x_1 = t^3$, $x_2 = t^2 - t$, $x_3 = 1$, (b) $x_1 = t^3$, $x_2 = t(1 - at^2)$, $x_3 = 1 - at^2$. Find also the involution $I_{2,1}$ of the contacts of each system with the base curve.

7. The rational quartic curve has ∞^1 perspective conics. Find the

equations of these conics for the quartic (*b*) Ex. 7, §160. Show that the conics have four contacts with the quartic which belong to an involution $I_{1,3}$ (pencil of binary quartics). Repeat for the curve $x_1 = at^3 + 1$, $x_2 = t^4 + at$, $x_3 = t^2$. When $a = 1$ the conic reduces to $u_3^2 = 0$, which is then a triple point of the quartic.

8. The rational quintic curve has a unique perspective conic. Find the perspective conic for the quintic $x_1 = t^5 + 2t^2$, $x_2 = -2t^3 - 1$, $x_3 = t^4 - t$. Also for the curve (*d*) Ex. 7, §160. Show that the perspective conic has five contacts with the quintic in each case.

174. The cuspidal cubic.—When the nodal parameters of a rational cubic coincide the node becomes a cusp. But it will be recalled that if a binary cubic have a double root the Hessian has the same double root and conversely. Consequently two flex parameters coincide with the cusp parameter, *i. e., two flexes fall together at the cusp.*[1]



Then assigning to the remaining point of inflexion the parameter 0, to the cusp the parameter ∞ and taking for sides of the triangle of reference the cusp tangent, the flex tangent and the line joining the cusp and flex *the equations of the cubic with a cusp can be written*

$$\begin{aligned} x_1 &= t^3 \\ x_2 &= t \\ x_3 &= 1. \end{aligned} \tag{1}$$

Eliminating t, the ternary equation is

$$x_2^3 - x_1x_3^2 = 0. \tag{2}$$

[1] This is an instance of the general fact that two flexes of a curve are absorbed when a double point is changed into a cusp.

The parameters of the intersections of the line $(ux) = 0$ with the curve are, substituting for the x's from (1)

$$u_1t^3 + u_2t + u_3 = 0. \tag{3}$$

If t_1, t_2, t_3 are the roots of this equation obviously

$$s_1 \equiv t_1 + t_2 + t_3 = 0 \tag{4}$$

which is *the necessary and sufficient condition that three points of the curve be collinear.*

Setting $t_1 = t_2 = \tau$ and $t_3 = t$ we have as the *relation connecting a point* τ *and its tangential* t

$$2\tau + t = 0. \tag{5}$$

Since (5) is linear in τ only one tangent can be drawn from an arbitrary point t of the curve, *i. e.*,

The cuspidal cubic is of class three.

The equations of the general rational cubic in the canonical form which we have adopted (§168) cannot be reduced to those of the cuspidal cubic (1) since there are no explicit constants to be made use of. But if we write the curve in the form

$$x_1 = t^3,\ x_2 = at^2 + t,\ x_3 = 1, \tag{6}$$

thus assigning to the flexes the parameters 0, ∞ and $-1/a$, and taking for triangle of reference two flex tangents and the line of flexes, the specialization is possible. For setting $a = 0$, two flexes coincide to form a cusp at $t = \infty$ and equations (6) reduce to (1).

EXERCISES

1. Find the parametric line equations and the ternary line equation of the cuspidal cubic.

2. Find the equation of the line joining two points t_1, t_2. What is the equation of the tangent at t? Write the equation of the tangent from t.

3. Obtain the condition that three points t_1, t_2, t_3 be the contacts of tangents from a point of the plane.

4. From the relation between a point and its tangential show that if we begin with any point on the cuspidal cubic and draw the tangent at the point, then draw the tangent at the tangential point and continue the process indefinitely the tangential point will approach the cusp as a limit. On the other hand if we draw the tangent *from* any point t, then draw the tangent from the point of contact and continue indefinitely the contact will approach the flex as a limit.

5. Find the equation of the satellite of a line with respect to the cuspidal cubic.

6. Show that the contacts of tangents from three collinear points on the cuspidal cubic lie on a line, the *primary* line of the first. Find the equation of the primary of a line u.

7. If a line is tangent to the cuspidal cubic, what can you say of the satellite line? the primary line?

8. There is a (1, 1) correspondence between the lines of a plane and (a) their satellites, (b) their primaries with respect to a cuspidal cubic. Write these correspondences as collineations (in line coördinates).

9. If a line envelops a curve then (a) its satellite, (b) its primary with respect to the cuspidal cubic envelops a curve of the same class. (Use the results of Ex. 8.)

10. The condition that six points of the cuspidal cubic (1), §174, lie on a conic is that the sum of their six parameters be zero. What is the condition that $3n$ points lie on a curve of order n?

11. Discuss the contact conics of the cuspidal cubic.

12. All cuspidal cubics are projectively equivalent. (*Cf.* Ex. 8, §35.)

13. Obtain the quadratic involution set up on the cuspidal cubic by the pencil of lines on a point t_1. Where are the double points?

14. Write the Hessian of the cuspidal cubic and examine its relation to the original curve.

15. Dualize Exs. 6–9.

16. Show that the cubic (2) §26 is cuspidal and that the conic osculates at two points. (Write the curve homogeneously and then find the parametric equations.)

175. Osculants of the rational cubic.—Another important class of curves associated with rational curves are the *osculants*. If the equation of a rational curve are

$$x_i' = f_i(t, \tau), \qquad i = 1, 2, 3, \quad (1)$$

where f_i are binary forms of order n in the homogeneous parameter t/τ, then the equations

$$x_i = \left(t_1\frac{\partial}{\partial t} + \tau_1\frac{\partial}{\partial \tau}\right)f_i \tag{2}$$

obtained by taking the first polars of f_i with respect to (t_1, τ_1) represent the *first osculant* of (1) at the point t_1/τ_1. Likewise, polarizing (2) with respect to t_2/τ_2 we obtain

$$x_i = \left(t_2\frac{\partial}{\partial t} + \tau_2\frac{\partial}{\partial \tau}\right)\left(t_1\frac{\partial}{\partial t} + \tau_1\frac{\partial}{\partial \tau}\right)f_i \tag{3}$$

which is a first osculant of (2) and a second osculant of (1). And so on until f_i are completely polarized. Some of the points t_i/τ_i may coincide when repeated polars result. Thus

$$x_i = \left(t_1\frac{\partial}{\partial t} + \tau_1\frac{\partial}{\partial \tau}\right)^r f_i \tag{4}$$

is the rth osculant at t_1/τ_1 of (1), the $(r-1)$ th osculant of (2), etc.

In practice after the equations of the osculants have been formed we may return to the non-homogeneous parameter by setting $\tau_i = \tau = 1$.

Obviously all osculant curves are rational.

We shall now consider briefly the osculants of the rational cubic. If we assign to the flexes the parameters $0, -1, \infty$ and select for triangle of reference the line of flexes and the tangents at the flexes 0 and ∞ the equations of the curve may be written in points

$$\begin{aligned} x_1 &= t^3 \\ x_2 &= 3t^2 + 3t \\ x_3 &= 1, \end{aligned} \tag{5}$$

and in lines (§128, (9))

$$\begin{aligned} u_1 &= 2t + 1 \\ u_2 &= -t^2 \\ u_3 &= t^4 + 2t^3. \end{aligned} \tag{6}$$

The osculant conic at t_1 is by (2)

$$\begin{aligned} x_1 &= t_1t^2 \\ x_2 &= t^2 + 2(1 + t_1)t + t_1 \\ x_3 &= 1 \end{aligned} \tag{7}$$

the ternary form of which in lines is (§ 123, (6))

$$u_2u_3 + t_1u_3u_1 + t_1^2u_1u_2 - (1 + t_1 + t_1^2)u_2^2 = 0. \tag{8}$$

The eight common lines of this conic and the cubic are, combining (6) and (8)

$$t^2\{t^4 + 2(1 - t_1)t^3 + (1 - 4t_1 + t_1^2)t^2 + 2t_1(t_1 - 1)t + t_1^2\} = 0. \tag{9}$$

Now this octavic equation contains the two flex parameters 0 and ∞ as double roots, hence the conic touches the corresponding flex lines. But there is no reason why the flex lines should not be treated impartially whence we infer that (9) contains the other flex parameter doubly. This is easily verified and (9) turns out to be

$$t^2(t + 1)^2(t - t_1)^2. \tag{10}$$

It appears now that the remaining roots of (10) are t_1 repeated, *i. e.*, the conic touches the curve at the point t_1. We have thus proved for the rational cubic a characteristic property of first osculants:

The first osculant at t_1 of the rational cubic is inscribed in the three flex tangents and touches the curve at t_1.

The second (linear) osculant at t_1, found at once (§96) by interchanging t and t_1 in (7) is

$$\begin{aligned} x_1 &= t_1^2t \\ x_2 &= (1 + 2t_1)t + 2t_1 + t_1^2 \\ x_3 &= 1. \end{aligned} \tag{11}$$

The ternary equation of this line is (§58, Ex. 16)

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ t_1^2 & 1 + 2t_1 & 0 \\ 0 & 2t_1 + t_1^2 & 1 \end{vmatrix} = 0$$

or

$$(1 + 2t_1)x_1 - t_1^2x_2 + (t_1^4 + 2t_1^3)x_3 = 0. \tag{12}$$

Referring to (6) it appears that (12) is simply the map equation of the cubic in lines with t_1 substituted for t, *i. e.*, (12) is the tangent at t_1. We have thus proved for the cubic a characteristic property of linear osculants: *The linear osculant at the point t_1 of the rational cubic is the tangent at t_1.*

EXERCISES

1. Find the first and second osculants of the rational cubic in the standard canonical form. Also of the curve $x_1 = t^3$, $x_2 = t(1 - at^2)$, $x_3 = 1 - at^2$.

2. The osculants at ω and at ω^2 of cubic (5) touch the line of flexes.

3. Find the osculant conic and the linear osculant of the cuspidal cubic, §174. What is the relation of the osculant conic to the cusp and the cusp tangent? Where are the 6 common lines?

4. The rational cubic is the envelope of osculant conics and also of linear osculants.

5. Show that the osculant conic of a point of inflexion on the rational cubic is the flex tangent repeated. The osculant conic of either nodal parameter degenerates,—into what?

6. Given three points t_1, t_2, t_3 on the rational cubic. Show that the mixed (linear) osculant of t_2, t_3 of t_3, t_1 and of t_1, t_2 meet in a point. (Morley, *On reflexive geometry*, Trans. Amer. Math. Soc., 1907, p. 16.)

7. The binary quartic giving the four points, in addition to the contact, in which an osculant conic cuts the rational cubic is self-apolar. (Thomsem, *The osculants of plane rational quartic curves*, Amer. Jour. Math., 1910.)

8. Dualize the theorems of this section. Also Exs. 5–7.

9. Extend the last theorem of this section to the rational quartic $x_1 = at^3 + 1$, $x_2 = t^4 + at$, $x_3 = t^2$.

10. Find the first osculants (osculant conics) of the cardioid, §168 considered as a class cubic. *Ans.* The line equations of the osculant at t_1 are $u_1 = -1$, $u_2 = t_1t^2$, $u_3 = -t^2 + 2(1 - t_1)t + t_1$.

The point equation in absolute coördinates is $x = t + t_1 - tt_1$.

11. The osculant conics of the cardioid (Ex. 10) are circles passing through the real cusp. The centers of these circles lie on another circle which passes through the cusp. (See Ex. 20, §156.) Thus given a circle C and a fixed point P on it, the envelope of the system of circles with centers on C and passing through P is a cardioid of which P is a cusp. Draw such a system of circles.

176. The rational cubic as a self-projective curve.— Taking the equations of the curve in the canonical form

$$x_1 = 3t^2, \qquad x_2 = 3t, \qquad x_3 = t^3 + 1 \tag{1}$$

or

$$x_1^3 + x_2^3 - 3x_1x_2x_3 = 0 \tag{2}$$

it appears that

The rational cubic is invariant under a dihedral collineation group of order 6.

For the generating transformations of the binary g_6

$$t' = \omega t, \qquad t' = 1/t, \qquad \omega^3 = 1, \tag{3}$$

when applied to the parameter in (1) set up the ternary collineations

$$\begin{aligned} x_1' &= \omega^2 x_1 & \qquad x_1' &= x_2 \\ x_2' &= \omega x_2 \quad \text{and} & x_2' &= x_1 \\ x_3' &= x_3 & x_3' &= x_3 \end{aligned} \tag{4}$$

which in turn generate a ternary dihedral G_6 which leaves the curve (2) unaltered. Observing that the reference triangle is the invariant triangle of the group we may say

The invariant triangle of the group consists of the nodal tangents and the line of flexes.

The special sets of parameters are (§159) (a) $2t = 0$, (b) $t^3 + 1 = 0$, (c) $t^3 - 1 = 0$. Of these (a) represents the nodal parameters $0, \infty$, (b) are the flex parameters and (c) are the parameters of the contacts of tangents from the flexes (169).

The special sets of the ternary group must either lie on the axes or at the centers of reflexion. Now the axes $x_1^3 - x_2^3 = 0$ meet at the node which is accordingly a fixed

point of the entire G_6. Substituting the t's from (1) in these equations we find for the intersections of the axes $t^3(t^3 - 1) = 0$, hence the points (c) lie on the axes and represent a special set of the ternary group. Finally the double points of the involutions must represent fixed points of the reflexions. Consequently they must lie either at the centers or on the axes of reflexion. We have just seen that three of the double points, *viz.*, $t^3 - 1 = 0$ lie on the axes and account for all of the intersections of the axes exclusive of the double point. Hence the remaining double points $t^3 + 1 = 0$ must fall at the centers of reflexion. Summarizing

The centers of reflexion are the points of inflexion and the axes are lines from the node to the contacts of tangents from the flexes.

We note also that the sextactic points $t^3 - 1 = 0$ lie on the axes and constitute a special set.

Thus the points of the curve which form special sets of the ternary group are (a) the node, (b) the three points of inflexion, (c) the intersections of the axes exclusive of the node. A simple construction for the ∞^1 general sets of conjugate points on the curve is afforded by the pencil of invariant conics

$$x_1x_2 - \lambda x_3^2 = 0. \tag{5}$$

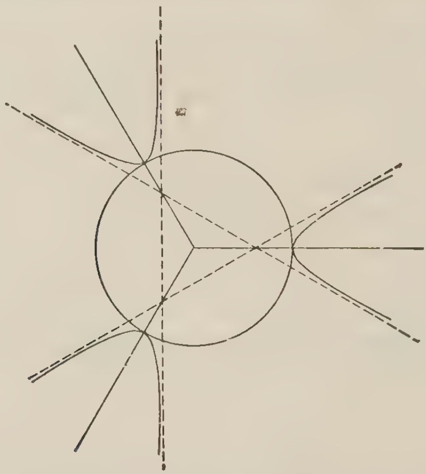
For since the conics and the cubic are both invariant under the group any intersection of the cubic with a conic of the pencil will be transformed by the group into a point that lies on both curves. Hence the conics of the pencil will ordinarily cut the cubic in general sets of 6 conjugate points. But the conics corresponding to $\lambda = 0, \infty$ and $9/4$ (conic N) cut out the points given by $t^3 = 0$, $(t^3 + 1)^2 = 0$, and $(t^3 - 1)^2 = 0$ which represent the special sets. Hence

Every member of the pencil of conics (5) *cuts the cubic in a set of points, binary and ternary, which form a conjugate set, special or general, under the binary or ternary group. Conversely every set of conjugate points, binary or ternary, special or general, is cut out by a member of the pencil of conics.*

In absolute coördinates the equations of the cubic are

$$x = \frac{3t^2}{t^3 + 1}, \quad \bar{x} = \frac{3t}{t^3 + 1}. \tag{6}$$

The double point is now isolated at the origin and the axes are three equispaced lines about which the curve is sym-



metrical. The flexes are real but lie on the line at infinity so that the flex tangents are asymptotes. The sextactic points, the contacts of N are also real. The invariant conics (5) become concentric circles each of which cuts the curve in 6 points which lie at the vertices of two concentric equilateral triangles. On account of the symmetry the curve is readily drawn and we append a figure together with the conic N.

The cuspidal cubic. When the equations of the cubic with a cusp are written in the canonical form

$$x_1 = t^3, \quad x_2 = t, \quad x_3 = 1 \tag{7}$$

or

$$x_2^3 - x_1 x_3^2 = 0 \tag{8}$$

the binary collineation $t' = \alpha t$ (α a parameter) operating on (7) induces the ternary collineation $x_1' = \alpha^3 x_1$, $x_2' = \alpha x_2$, $x_3' = x_3$ which transforms (8) into itself. Hence

The cuspidal cubic is invariant under a one-parameter group. The binary group has two fixed points, the cusp and flex parameters, while the ternary group has a fixed triangle, the reference triangle.

The cuspidal cubic is also self-dual, being auto-polar with respect to $x_1^2 + 3x_2^2 + 2x_3^2 = 0$.

EXERCISES

1. The axes of reflexion of the rational cubic are the harmonic polars of the curve (Ex. 17, §168).

2. The six points associated with a set of osculating conics (Ex. 8, §172) are a general set of conjugate points of the G_6. Hence they lie on a conic of the invariant pencil. They form a Pascal hexagon whose opposite sides meet at the flexes.

3. The sextactic points lie on the axes of reflexion. Connect this with Ex. 2.

4. If C represents a rational cubic and H its Hessian, show that every member of the *syzygetic pencil* $C + \lambda H = 0$ is a rational cubic invariant under a G_6. (The pencil can be written in the form $x_1^3 + x_2^3 + kx_1x_2x_3 = 0$.) What are the degenerate members of the pencil?

5. The deltoid (§168) is invariant under the G_6 (in absolute coördinates, §159). The cardioid being projectively equivalent must admit a G_6 also. Find the transformations, binary and ternary, of the group of the cardioid.

6. The triangle of asymptotes of (6) is inscribed in the unit circle.

7. One circle of the pencil $x\bar{x} = k^2$ cuts (6) in the vertices of a regular 9-gon, omitting every third vertex. Find the radius of this circle.

8. Any curve of the form $x_2^n - x_1^a x_3^b = 0$, $(a + b = n)$, is invariant under a one-parameter group. Write the transformations of the group.

CHAPTER XIII

NON-EUCLIDEAN GEOMETRY

177. Measurement in Euclidean geometry.—Any metric geometry depends upon two fundamental processes,—the measurement of distances and angles. In the practical measuring of linear distances we use a graduated rule or linear scale which is applied repeatedly to the distance to be measured. Now the measure of a given distance is independent of the part of the scale used in the process. In other words the measure of the distance is unaltered by a sliding of the scale along itself. But this sliding of the scale along itself corresponds in the abstract to a translation (§92) which is expressed analytically by the equation

$$x' = x + k.$$

A translation in turn is the metrically canonical form of a parabolic collineation whose fixed points coincide at $x = \infty$. Thus one point (two coincident points) on every line, the point at infinity, remains fixed in the measuring operation. Or we may say that *the measure of distances in Euclidean (plane) geometry is parabolic.*

On the other hand the successive application of a unit angle to any given angle amounts to a rotation of the angle scale about the vertex of the angle to be measured. Now two distinct but conjugate imaginary lines on the center, namely the circular rays from the point, are fixed under a rotation of the plane (§161). Hence the rotation effects on the lines of the pencil a one-dimensional elliptic collineation, or *the measure of angles in Euclidean geometry is elliptic.*

178. The Cayley-Klein generalization of distance.—In the foregoing we have treated distance, in conformity with the practice in elementary geometry, as a primitive undefined concept. Among the properties commonly attributed to distances along a line are

1°. The additive property, *i. e.*,

$$\text{dist } ab + \text{dist } bc = \text{dist } ac.$$

2°. The distance from a point to itself is zero.

3°. The distance between two points is unaltered by a translation of the line.

The parabolic measure of distance is then a consequence of 3°. We may however establish an elliptic or hyperbolic scale along the line by a suitable extension of the notion of distance,—an extension suggested by Laguerre's theorem (§56): The angle formed by two lines of a pencil is equal to a certain multiple of the logarithm of the double ratio determined by the sides of the angle and the circular rays on the vertex. But the circular rays are the *fundamental* (fixed) lines under a rotation of the pencil which accompanies the angle measurement.

Returning to the line let us choose two fundamental points, with coördinates[1] p, q, which shall be the fixed points of a non-singular collineation of the line. Then if x, y are the coördinates of any other two points of the line we shall define their distance thus

$$\text{dist } xy = k \log (xy|pq)$$

where k is a fixed but arbitrarily chosen constant.

This definition satisfies the requirements 1° and 2°

[1] We assume here with Cayley (*A Sixth Memoir upon Quantics*, *Collected Papers*, Vol. 2, p. 605) that coördinate is our undefined concept, that there is a (1, 1) correspondence between the points of the line and the number system and that the number corresponding to any point is the coördinate of the point. The numbers x, y, etc. need not represent distances in the ordinary sense.

above. Thus if x, y, z be any three points on the line then $(xy|pq)(yz|pq) = (xz|pq)$. Hence taking logarithms

$$k \log (xy|pq) + k \log (yz|pq) = k \log (xz|pq)$$

i. e.,

$$\text{dist } xy + \text{dist } yz = \text{dist } xz,$$

which proves 1°. The proof of 2° is immediate for

$$k \log (xx|pq) = k \log 1 = 0.$$

The definition also satisfies a requirement of which 3° is a special case, *viz.*, the distance between two points is unaltered by a collineation which leaves p and q fixed. This follows at once from the invariant property of a double ratio.

Moreover the new distance may be reduced to the Euclidean notion by appropriate specialization when the two fundamental points coincide at infinity. If $p \equiv q$, dist $xy = 0$ $(x,\ y \neq p)$ so long as k is finite. But if k becomes infinite the distance takes the indeterminate form $\infty \cdot 0$. We may then assign a value to the distance by a limit process.

Let $q = p + \epsilon$ and $k = k'/\epsilon$ where ϵ is infinitesimal. We define dist xy as follows:

$$\text{dist } xy = \lim_{\epsilon \to 0} \frac{k'}{\epsilon} \log (xy|p\,\overline{p + \epsilon}).$$

We have

$$(xy|p\,\overline{p + \epsilon}) = \frac{(x-p)(y-p-\epsilon)}{(x-p-\epsilon)(y-p)} =$$

$$\left(1 - \frac{\epsilon}{y-p}\right)\left(1 - \frac{\epsilon}{x-p}\right)^{-1} =$$

$$\left(1 - \frac{\epsilon}{y-p}\right)\left(1 + \frac{\epsilon}{x-p} + \frac{\epsilon^2}{(x-p)^2} + \ .\ .\ .\ \right)$$

Then neglecting the second and higher powers of ϵ which have no effect on the limit this reduces to

$$1 + \epsilon\left(\frac{1}{x-p} - \frac{1}{y-p}\right) = 1 + \epsilon f,$$

substituting f for the expression in parentheses. Now by McLaurin's formula

$$\log(1 + \epsilon f) = \epsilon f - \epsilon^2 f^2 + \epsilon^3 f^3 - \ . \ . \ .$$

Hence

$$\text{dist } xy = \lim_{\epsilon \to 0} \frac{k'}{\epsilon}(\epsilon f - \epsilon^2 f^2 + \ . \ . \ . \ ) =$$

$$k'f = k'\left(\frac{1}{x-p} - \frac{1}{y-p}\right) = k'\frac{y-x}{(x-p)(y-p)}.$$

Now choose a point e such that dist $ye = 1$. This requires

$$k' = \frac{(e-p)(y-p)}{e-y}.$$

We have then

$$\text{dist } yx = \frac{(x-y)(e-p)}{(x-p)(e-y)} = (xe|yp).$$

Finally choosing y, p as base points and e as unit point in a Cartesian system of coördinates we obtain, setting $y = 0$, $p = \infty$, $e = 1$

$$\text{dist } 0x = (x1|0\infty) = x$$

where the distance now has the usual Euclidean significance.

Q. E. D.

As a definition the measure of distance on the line is said to be of the same variety as the collineation that leaves the distance of two points invariant, *i. e.*, the measure of distance is *hyperbolic*, *parabolic* or *elliptic* according as the fundamental points are real and distinct, real and coincident or conjugate imaginary. Dually the measure of angle in the pencil will be hyperbolic, parabolic or elliptic when the fundamental lines are a hyperbolic, a parabolic or an elliptic pair.

179. Thus in the one-dimensional domain of the line or pencil we have *three varieties of metric geometry* corresponding to the three kinds of measure of the respective element distance and angle. After Klein[1] we shall call these geometries *hyperbolic, parabolic* and *elliptic.* The parabolic geometry is, as we have just seen, Euclidean. We shall now examine more in detail the nature of the line in each geometry,—the results can be translated at once into the geometry of the pencil by duality. In the discussion we suppose that x and y are real.

The hyperbolic line. *p, q real and distinct, k real* and finite. Now

$$k \log (xp|pq) = k \log \infty = \infty$$

and

$$k \log (xq|pq) = k \log 0 = -\infty,$$

i. e., the distance from any point of the line to a fundamental point is infinite, or

The hyperbolic line contains two real "points at infinity."

If x and y both lie between p and q, $(xy|pq)$ is positive and the distance is real. If one of the points (x or y) is between p and q while the other is not, $(xy|pq)$ is negative and the distance is imaginary. If x and y both lie without the segment pq, $(xy|pq)$ is again positive and the distance is real. Or dist xy *is real when the points x, y are not separated by p and q, imaginary when they are so separated.*

We have thus three classes of points on the hyperbolic line, *ordinary* points lying between p and q, points at infinity (p and q themselves) and points lying outside the segment pq, called *ultra-infinite* or *ideal* points. p and q thus divide the ordinary points from the ideal. Ideal points like ordinary may be either real or imaginary.

On the *parabolic* or *Euclidean line, k* is infinite and p and

[1] *Über die sogennante Nicht-Euklidische Geometrie, Mathematische Annalen,* Vol. 4.

q coincide at infinity. While the line is of infinite extent, the two ends come together at infinity. The parabolic line differs from the hyperbolic in having no ideal (ultra-infinite) points.

The elliptic line. *p and q conjugate imaginary.* Since the fundamental points are imaginary we can write $(xy|pq) = e^{i\theta}$. This is done exactly as in §56 by expressing the double ratio in terms of the parameters of the four points, using $\tan\theta$ as a parameter. Hence dist $xy = k \log (xy|pq) = ki\theta$. In order that the distance between two real points shall be real we must take k to be a pure imaginary, say ik'. We have then for the elliptic case

$$\text{dist } xy = ik' \log (xy|pq)$$

where k' is real.

The elliptic line has two conjugate imaginary points at infinity, namely p and q. Hence *the (real) elliptic line is finite in length* and contains ordinary points only.

The elliptic line is however *unbounded* or *closed, i. e.*, the "two ends" of the line are regarded as falling together.[1] Thus a point moving along the line in the same direction will, after traversing a finite distance, return to the starting point. The total length of the line is an absolute constant and supplies therefore a natural unit of distance.[2] The situation is analogous to that in the pencil in Euclidean geometry where the measure of the angle is elliptic. A line revolving through a finite angle returns to the initial position. The total angle generated is an absolute constant (2π or π according as the line is directed or not) and furnishes us our natural unit of angle.

Ex. Dualize this section.

[1] *Cf.* the parabolic line whose two ends however coincide at infinity. Moreover it is not possible by continuous motion in one direction to traverse the entire line.

[2] This is in sharp contrast with the Euclidean case where the unit of distance is arbitrary.

180. Metric geometry in the plane.—We saw that from the present point of view there are but three varieties of metric geometry in one dimension, for the geometry of the line and the pencil are abstractly identical. In the plane however we require both a measure of distance and of angle and since we have three choices for each we are led to nine species of metric geometry. Tradition has however fixed the measure of angle to be elliptic as in Euclid. With this limitation we have again three types of metric plane geometry,—hyperbolic, parabolic and elliptic, corresponding to the three methods of measuring distance.

The parabolic geometry in the plane is Euclidean. The others which contradict Euclid in essential particulars are known as the *hyperbolic* and *elliptic non-Euclidean* geometries. The relation between the non-Euclidean geometries and Euclid on the one hand and between the elliptic and hyperbolic geometries themselves on the other is most clearly revealed by an examination of the *absolute* or infinite region. When we first considered infinite elements (§20) it will be recalled that we *postulated* one point at infinity on a line, *i. e.*, we deliberately chose the parabolic line.[1] The result was a *line* at infinity in the plane. More properly the absolute in Euclidean geometry consists of a point pair,—the circular points. It is therefore a degenerate line conic which as a point conic appears as a (repeated) line. We might however with equal propriety postulate the hyperbolic or elliptic line. For we are free to select any locus for the infinite region (absolute) so long as our choice does not conflict with axioms previously adopted.[2]

[1] This was because we wished to exhibit the relation between projective geometry and the familiar Euclidean.

[2] It is presumed that the student is acquainted with the modern view, that axioms are not self-evident truths but basic propositions incapable of proof or which are to be accepted without proof. In other words axioms (or postulates—we make no distinction here,) are simply a part of the hypothesis of every subsequent proposition—though of course every

The absolute would then consist of a proper conic, real or imaginary.

Throughout this course we have seen that Euclidean theorems can be derived from projective by isolating a pair of points for the circular points or a line for the line at infinity,—that the peculiar features of Euclidean geometry have their roots in the nature of the absolute. In precisely the same way by isolating a real conic for the absolute we should obtain a hyperbolic theorem as a special case of a projective theorem. Likewise by isolating an imaginary conic for the absolute a projective theorem is specialized to an elliptic. Hence *metric geometry is simply projective geometry with special relation to an absolute conic.* The metric geometry is hyperbolic or elliptic non-Euclidean according as the absolute conic is real or imaginary; it is Euclidean when the absolute degenerates to a pair of points.

It appears that Euclidean geometry is the limiting case of both non-Euclidean geometries, occupying a position between the two. For in the continuous metamorphosis of a real proper conic into an imaginary conic the conic must assume a degenerate form which divides the real conics from the imaginary.[1]

Again any projective transformation that leaves the

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axiom need not be used in the proof of every proposition. A theorem of geometry is a logical inference from the axioms (or other propositions deduced from them). The only logical requirement of a system of axioms is that they be consistent, *i. e.*, that they do not contradict one another. We may be governed otherwise in our choice by convenience, personal interest or wholly by caprice. But no important development is likely to follow from a system of axioms chosen without regard to their plausibility, general appeal or ready adaptability to an extensive field of thought or phenomena. Thus Euclid's "common notions" and postulates were probably selected because they conformed to his intuitive notions of the world of space. And the fifth postulate was attacked because it failed to satisfy the demands of his followers as a proposition acceptable without proof.

[1] *Cf.* the case of the circle (§ 35) which is however the dual of the present instance.

absolute unaltered will not disturb the character of the metric theorem. Accordingly the hyperbolic or elliptic geometry consists of those properties which are invariant under a projective transformation that leaves the associated absolute conic fixed.[1]

Hyperbolic Geometry

181. Classification of points and lines.—In hyperbolic geometry the absolute conic divides the real points of the plane into two classes,—ordinary or *actual* points lying within and *ideal* points lying without the conic.[2] The points of the absolute itself are *points at infinity* and constitute the locus of such points.

Likewise there are three classes of lines,—*actual* lines which cut the absolute in two real (distinct) points, *isotropic* lines which are tangent to the absolute and *ideal* lines lying wholly in the ideal region of the plane and which cut the absolute in conjugate imaginary points.

Again pairs of actual lines are divided into three classes by definition:

(1) *intersectors*, which meet in actual points
(2) *parallels*, which meet in points at infinity
(3) *non-intersectors*, which meet in ideal points.

An immediate consequence of the definition (2) is

Through any point P two lines can be drawn parallel to a given line l.

For l cuts the absolute in two points and each point determines with P a line parallel to l. When l is an actual line the intersectors of l in the pencil on P are separated from the non-intersectors by the two parallels.

[1] A similar statement does not hold for Euclidean geometry. For the circular points are invariant under the equi-form group which contains 4 parameters whereas the group of displacements is a 3-parameter group.

[2] For the distinction between inside and outside see §71.

182. Angle between two lines.—Since the measure of angle is to be elliptic (§180) we may appropriate the definitions and conventions already employed in metric geometry:

The angle between two lines is a constant multiple of the logarithm of the double ratio formed by the lines and the tangents to the absolute from their intersection, i. e.,

$$\text{ang } uv = \mu \log (uv|\alpha\alpha') \tag{1}$$

where α and α' are isotropic lines concurrent with u and v.

Again, *lines which are conjugate with respect to the absolute are perpendicular.* If now we adopt such a unit of angle that the measure of a right angle is $\pi/2$ as in elementary trigonometry the constant μ will be fixed.[1] For if u and v are perpendicular, $(uv|\alpha\alpha') = -1$ (by definition). And since $\log(-1) = i\pi$ §58, Ex. 8) we have from (1) if ang $uv = \pi/2$

$$\pi/2 = \mu\, i\pi \text{ or } \mu = 1/2i.$$

Theorem. All lines which are perpendicular to a given line l meet in a point P, the pole of the line with respect to the absolute conic. Conversely *all lines on a point P are perpendicular to a unique line l, the absolute polar of P.*

This follows at once from the definitions. If P is actual, l is ideal; if P is at infinity, l is isotropic (with contact at P); while if P is ideal, l is actual. Thus two lines always have a unique perpendicular. When the lines are parallel the common perpendicular is tangent to the absolute at their point of intersection.[2]

[1] The unit angle however is not the radian as defined in trigonometry, *i. e.*, the angle at the center of a circle subtended by an arc equal in length to the radius for in non-Euclidean geometry this angle is not the same for all circles.

[2] Contrast this with Euclid where parallel lines have an infinity of common perpendiculars.

EXERCISES

1. Two ideal points may determine an actual, an isotropic or an ideal line.

2. Two lines each of which is parallel to a third line in the same sense are parallel to each other. (This theorem holds either in the plane or space.)

3. Two parallel lines meet at a zero angle. Hence the sum of the angles of a triangle inscribed in the absolute is zero. (This is a triangle of maximum area.)

4. The angle formed by two actual lines is real or imaginary according as the lines are intersectors or non-intersectors. The angle between two ideal lines is imaginary. The angle between an isotropic line and a non-isotropic line (actual or ideal) is ∞ unless indeed the lines meet on the absolute when the angle is indeterminate (being both 0 and ∞). The angle between two isotropic lines is ∞ (by a limit process).

5. Two lines which are perpendicular to the same (non-isotropic) line are not parallel.

6. A (non-isotropic) line which is perpendicular to one of two parallel lines is not perpendicular to the other.

7. An isotropic line is perpendicular to itself.

8. A unique perpendicular can be drawn through a point to a line,—unless the line is isotropic and the point is the point of contact when the perpendicular is indeterminate.

9. A self-polar triangle is tri-rectangular. In hyperbolic geometry two vertices are ideal.

10. If one of two points which are conjugate to the absolute is actual, the other is ideal. Is the converse true?

11. If the equation of the absolute is $x^2 + y^2 - k^2z^2 = 0$ the condition that the lines $a_1x + a_2y + a_3z = 0$ and $b_1x + b_2y + b_3z = 0$ be perpendicular is $a_1b_1 + a_2b_2 - a_3b_3/k^2 = 0$.

12. If the equation of the absolute is $\alpha_1x_1^2 + \alpha_2x_2^2 + \alpha_3x_3^2 = 0$, the equation of the line through the point (a_1, a_2, a_3) and perpendicular to the line (u_1, u_2, u_3) is

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ a_1 & a_2 & a_3 \\ \dfrac{u_1}{\alpha_1} & \dfrac{u_2}{\alpha_2} & \dfrac{u_3}{\alpha_3} \end{vmatrix} = 0.$$

13. Find the condition that two lines be parallel (intersect on the absolute). (Take the lines as in Ex. 11 and the absolute as in Ex. 12.)

If $\alpha_i = 1$, the condition can be thrown into the form $(aa)(bb) - (ab)^2 = 0$, where $(ab) \equiv a_1b_1 + a_2b_2 + a_3b_3$.

183. The distance between two points x, y *is defined to be a constant multiple of the logarithm of the double ratio of the points and the two points in which their junction cuts the absolute, i. e.,*

$$\text{dist } xy = \frac{k}{2} \log (xy|aa') \tag{1}$$

where a and a' are the absolute points collinear with x and y. We have thus a sort of duality between distance and angle in virtue of the similarity of the numerical measures.[1]

The distance between two points conjugate with respect to the absolute is called a quadrant. If x and y are conjugate points we have $(xy|aa') = -1$, hence

$$\text{quadrant} = \frac{k}{2} \log (-1) = \frac{k\pi i}{2}. \tag{2}$$

But if x is fixed y may be anywhere on the polar of x, *i. e.*,

The locus of points a quadrant distant from a fixed point x is the absolute polar of x.

If two distances whose sum is a quadrant are called *complementary* we may say

The distance from a point to a line is the complement of the distance from the point to the absolute pole of the line.

The distance between two points is proportional to the angle between their absolute polars or the angle between two lines is proportional to the distance between their absolute poles.

For the polars of x, y, a, a' form a pencil u, v, α, α' which is projective with the range (§74, (3)), hence $(xy|aa') = (uv|\alpha\alpha') = r$. Therefore

$$\text{dist } xy = \frac{k}{2} \log r = ik \text{ ang } uv. \tag{3}$$

[1] We also see why duality fails in Euclid for the absolute there is not dually specialized consisting as a line locus of two distinct points but as a point locus of a repeated line.

The distance of two points can be otherwise expressed. Let the equation of the absolute conic A be

$$A = f(x) = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{23}x_2x_3 + 2a_{31}x_3x_1 + 2\,a_{12}x_1x_2 = 0$$

and let A_{xy} denote the polarized form of A, *i.e.*,

$$A_{xy} = \frac{1}{2}\left(y_1\frac{\partial}{\partial x_1} + y_2\frac{\partial}{\partial x_2} + y_3\frac{\partial}{\partial x_3}\right)f(x).$$

Then

$$A_{xx} = A = f(x) \text{ and } A_{yy} = f(y).$$

Now choosing x and y as base points in a parametric representation along the line exactly as in §135 we have

$$f(tx + y) = A_{xx}t^2 + 2A_{xy}t + A_{yy} = 0 \tag{4}$$

the roots of which are the parameters t_1, t_2 of the absolute points of the line xy. But since the double ratio of four points is equal to the double ratio of their parameters (§55) we have $(xy|aa') = (0\infty|t_1t_2) = t_1/t_2$. Hence substituting the values of t_1 and t_2 from (4)

$$\text{dist}\ xy = \frac{k}{2}\log(t_1/t_2) = \frac{k}{2}\log\frac{A_{xy} + \sqrt{A_{xy}^2 - A_{xx}A_{yy}}}{A_{xy} - \sqrt{A_{xy}^2 - A_{xx}A_{yy}}} \tag{5}$$

which is Klein's formula for the distance.

We can also write (5) in a trigonometric form through the introduction of hyperbolic functions. We make use of the relation[1]

$$\cos ix = \cos(-ix) = \frac{e^x + e^{-x}}{2} = \text{ch}\ x = \text{ch}\ (-x)$$

or

$$-ix = \text{arc cos}\ \frac{e^x + e^{-x}}{2}. \tag{6}$$

Then replacing e^x by $\sqrt{u}$ and x by $\frac{1}{2}\log u$ we obtain

$$\log u = 2i\ \text{arc cos}\ \frac{u+1}{2\sqrt{u}}. \tag{7}$$

[1] See Hulburt, *Calculus*, p. 392.

Thus with the aid of (7), (5) takes the form

$$\text{dist } xy = \frac{k}{2} \log \frac{t_1}{t^2} = ik \text{ arc cos } \frac{t_1 + t_2}{2\sqrt{t_1 t_2}}. \tag{8}$$

But

$$t_1 + t_2 = -2A_{xy}/A_{xx} \text{ and } t_1 t_2 = A_{yy}/A_{xx} \qquad (\text{by } (4)).$$

Hence, choosing the negative sign of the radical, (8) becomes

$$\text{dist } xy = ik \text{ arc cos } \frac{A_{xy}}{\sqrt{A_{xx}A_{yy}}}, \tag{9}$$

which is Cayley's distance function.

If we refer the absolute to a self-polar triangle its equation reduces to a sum of squares (§140)

$$x^2 + y^2 - k^2z^2 = 0 \tag{10}$$

where the constant is introduced in a form to insure the reality of the conic. The distance between the points (x, y, z) and (x', y', z') is then, when written in full

$$\text{dist } xx' = ik \text{ arc cos } \frac{(xx' + yy' - k^2zz')}{\sqrt{(x^2 + y^2 - k^2z^2)(x'^2 + y'^2 - k^2z'^2)}}. \tag{11}$$

EXERCISES

1. The distance between two points on an isotropic line is zero. If the distance between two points is zero, either the points coincide or they lie on an isotropic line.

2. Prove the following properties of distance between real points in the hyperbolic plane: If the points are actual the distance is real. If one point is actual and the other a point at infinity, the distance is infinite. If the points are ideal the distance is real or imaginary according as the line which they determine is actual or ideal. If one point is at infinity and the other is ideal the distance is infinite unless the junction of the points is isotropic when the distance is indeterminate.

3. Every line segment has two middle points, *viz.*, the double points of the involution determined by the end points of the segment

and the absolute points collinear with them. (Let p and q be the end points of the segment and a, a' be the absolute points collinear with p, q. Assign parameters to the points along the line respectively t_1, t_2, 0, ∞. Then the parameters of the double points c_1, c_2 of the involution are $\pm\sqrt{t_1t_2}$. Hence show that $pc_i = c_iq$.)

4. A triangle has six medians (Ex. 3) which meet by threes in four points.

5. Write the Cayley and the Klein formula for the distance between two points in one-dimensional hyperbolic geometry when the equation of the fundamental (absolute) points is $A \equiv ax_1^2 + 2bx_1x_2 + cx_2^2 = 0$. *Ans.* Same as in two dimensions where, however,

$$A_{xy} \equiv \frac{1}{2}\left(y_1\frac{\partial}{\partial x_1} + y_2\frac{\partial}{\partial x_2}\right)A.$$

6. Write the Cayley and the Klein formula for the angle between two lines when the absolute is (a) the general conic, (b) the special conic of this section.

184. The circle.—If we define a circle as the locus of a point at a constant distance from a fixed point, the distance formulas of the last section furnish us at once the equation. For we have only to consider one of the points as fixed. Thus using Cayley's distance function, the equation of a circle with center (c_1, c_2, c_3) and radius r is

$$\cos\frac{r}{ik}\left(=\operatorname{ch}\frac{r}{k}\right) = \frac{A_{cx}}{\sqrt{A_{cc}A_{xx}}} \tag{1}$$

or

$$C\colon A_{cc}A_{xx}\operatorname{ch}^2\frac{r}{k} - A_{cx}^2 = 0. \tag{2}$$

But this equation can be combined with the absolute to form the square of a linear equation, in fact

$$\lambda A_{xx} - C \equiv A_{cx}^2, \text{ where } \lambda = A_{cc}\operatorname{ch}^2\frac{r}{k}. \tag{3}$$

Hence (§140) the circle touches the absolute at the points in which the line $A_{cx} = 0$ cuts. Calling this line, which is the absolute polar of the center, the *axis* of the circle we may say

A circle is a conic which has double contact with the absolute, whose axis is the chord of contact and whose center is the pole of the axis with respect to either the absolute or the circle itself.

Dually *the envelope of a line cutting a fixed line at a constant angle is a circle of which the fixed line is the axis.*

Since every point of the axis is a quadrant distant from the center, *a circle is also the locus of a point at a constant distance, namely the complement of the radius, from the axis.*[1]

When the axis is an actual line the circle appears (in the finite region) as a bipartite curve symmetrical with respect to the axis. It is then called an *equidistant* curve.

It follows from Ex. 1 (dual) §79 that if two conics have double contact, a tangent to one conic and the line joining the contact of the tangent to the pole of their common chord are a pair of conjugate lines with respect to the other conic. When one conic is the absolute we get at once: *A tangent to a circle is perpendicular to the radius drawn to the point of contact, i. e., a circle is the orthogonal trajectory of a pencil of lines.*

185. Classification of circles.—The aspect of a circle depends of course on its behavior at infinity. There is an essential distinction according as the axis is an actual or an ideal line for the contacts with the absolute are then respectively real or imaginary. In the first case we have the equidistant curve. In the second the (real) points of the circle form a closed curve, lying wholly in the finite region and the circle resembles a Euclidean circle. This curve is commonly called a *proper* circle.[2]

[1] It follows of course that the locus of a point at a constant distance from a line cannot be two parallel lines as in Euclid. In other words *parallel lines in hyperbolic geometry are not equidistant.* And equidistant (straight) lines do not exist. Non-Euclidean geometry proves that *equidistance* and *parallelism*, which are frequently confused, are essentially independent ideas.

[2] The equidistant curve is likewise proper, *i. e.*, non-degenerate.

Suppose now that the center of a proper circle move off to infinity. The radii tend to become parallel lines and the axis approaches an isotropic line. The two points of intersection of the axis with the absolute, *i. e.*, the two contacts of the curves, approach coincidence. Thus the limit of the circle as the center recedes to infinity is a conic having 4-point contact with the absolute. It is called a *horocycle*, the point of contact being the center and the tangent at the point the axis. Likewise starting with an equidistant curve we may suppose the axis to recede to infinity when the limit of the curve would again be a horocycle. Thus beginning with a proper circle with actual center and ideal axis, as the center passes from an actual through an infinite to an ideal point, the axis will change respectively from an ideal through an isotropic to an actual line and the circle is continuously transformed from a proper circle through a horocycle to an equidistant curve.[1]

Since the distance from any point in any direction to the absolute is infinite, the absolute itself may be considered a circle with arbitrary center and infinite radius.

Coming now to degenerate circles, it is clear geometrically that a pair of isotropic lines fulfills the condition for a circle, having two contacts with the absolute. If the lines are an elliptic pair they will meet in a real (and actual) point and the contacts will be conjugate imaginary. We thus have a *null* circle which is a degenerate case of proper circle. If the lines are a hyperbolic pair the contacts will be real but the circle will contain no actual points. This is a degenerate equidistant curve. In either case the center is the intersection of the line pair and the radius is zero. For

[1] The horocycle is thus the limiting form of both the proper circle and the equidistant curve and is sometimes called in consequence a *boundary* curve. *Cf.* the relationship of the conics in ordinary geometry. If the eccentricity approaches 1 from either direction, the ellipse or hyperbola approaches the parabola as a limit.

the pair of tangents from the point (c_1, c_2, c_3) to the absolute is (§135, (6)) the discriminant of the t-equation (4) §183, *viz.*,

$$A_{cc}A_{xx} - A_{cx}^2 = 0.$$

But since ch $0 = 1$, this is just the form assumed by (2) §184 when $r = 0$.

If the lines are coincident we get a degenerate horocycle with center at the contact of the line. The radius is however arbitrary since the distance from a point on an isotropic line to the point of contact is indeterminate.

We have already noted §183 that the locus of a point at a quadrant's distance from a point is a line. Setting $r = ki\pi/2$ in the equation of the circle (2) §184 we get $A_{cx}^2 = 0$, hence the circle is the polar line of the center, repeated. We distinguish two cases according as the line cuts the absolute in real or imaginary points.[1] Thus *an arbitrary line repeated is a circle with radius = a quadrant,* unless indeed the line is isotropic when the radius is arbitrary. If the line is ideal the circle is represented in the finite region only by its center which is the pole of the line.

We thus recognize nine types of circles which may be characterized as follows:

Species of Circles in Hyperbolic Geometry[2]

Non-degenerate

1°. *Proper circle;* center actual, axis ideal.

2°. *Horocycle;* center and axis incident at infinity, radius infinite.

3°. *Equidistant curve;* center ideal, axis actual.

4°. The *absolute;* center arbitrary, radius infinite.

[1] The case of two real coincident intersections has already been noticed under the name of degenerate horocycle.

[2] We take account only of equations of the form of (2) of the last section with *real* coefficients. This excludes the possibility of a degenerate circle consisting of one real and one imaginary isotropic line.

Degenerate

5°. *Imaginary null circle;* an elliptic pair of isotropic lines, radius zero (degenerate case of proper circle).

6°. *Real null circle;* a hyperbolic pair of isotropic lines, radius zero (degenerate equidistant curve).

7°. *Degenerate horocycle;* an isotropic line repeated, radius arbitrary.

8°. Actual line repeated, radius a quadrant.

9°. Ideal line repeated, radius a quadrant.

EXERCISES

1. A circle is determined by the center (or axis) and one point.

2. Concentric circles have the same axis and conversely. Hence concentric circles have double contact with each other at their absolute points.

3. Write the equation of a circle in full when the absolute is (a) the general conic, (b) the conic $x^2 + y^2 + z^2 = 0$, (c) $2yz + x^2 = 0$.

4. If the equation of the absolute is $x^2 + y^2 - k^2z^2 = 0$, the equation of a horocycle may be written $x^2 + y^2 - k^2z^2 = \lambda(ax + by + cz)^2$ where $a^2 + b^2 = c^2/k^2$. (The center is on the absolute and the axis (polar of the center) is an isotropic line.)

5. The polar reciprocal of a circle with respect to the absolute is a circle.

6. Write the equation of the null circle on the point (a_1, a_2, a_3). (Take the absolute as in Ex. 4.)

7. The envelope of a line which is perpendicular to a fixed line is a degenerate circle.

8. In the geometry along a line any point pair may be considered a "circle" whose center and "axis" are the double points of the involution determined by the point pair and the absolute points.

9. Write the equation of a circle considered as an envelope of a line cutting a fixed line at a constant angle when the absolute is taken (a) in the general form, (b) in the special form of §183.

186. Remarks on elementary hyperbolic geometry.—While the introduction of infinite and ideal elements in hyperbolic geometry effects an obvious economy of lan-

guage and makes clear the bond of union between projective and hyperbolic geometries[1] it should be said that *elementary hyperbolic geometry is concerned only with real points within the absolute conic, i. e.*, only actual points and lines are considered.[2] This would entail some revision and even elimination of certain theorems of the preceding sections but would involve little difficulty. Thus "imaginary point," "point at infinity," "ideal point" would alike be replaced by "no point." We should then have two classes of line pairs without a common point,—the parallels and the non-intersectors. Parallel lines might then be defined as the two lines of a pencil which separate the intersectors of a given line from the non-intersectors. The theorem that two lines which are perpendicular to the same line meet in an ideal point would be replaced by: Two lines which are perpendicular to the same line do not meet, neither are they parallel. A proper circle would have a center but no axis, a horocycle would have neither while an equidistant curve would not be considered a circle,—having only an axis. And so on. The reverse process however of recovering the projective from the elementary form of statement would not be so easy nor indeed always possible.

Elliptic Geometry

187. Many of the theorems of hyperbolic geometry carry over without change to elliptic geometry for each is projective geometry in relation to a proper absolute conic. There is however a striking difference in the real domain since many figures which are real in one become imaginary

[1] *Cf.* the utility of infinite elements in Euclidean metric geometry as exemplified throughout the book.

[2] Just so elementary Euclidean geometry considers the projective plane exclusive of one line, the line at infinity. The student will realize that the whole of trigonometry and most of elementary analytic geometry and calculus as treated in college courses are Euclidean.

in the other. And it is precisely this difference between real and imaginary that constitutes the true basis of distinction. For example in hyperbolic geometry two real lines can be drawn through a point parallel to a given line. But in elliptic geometry these two lines are imaginary since the points at infinity on any line are imaginary, in fact *there are no (real) parallel lines whatever.*

The measure of angle is the same as in Euclidean and hyperbolic geometry, *viz.*

$$\text{Ang } uv = \frac{1}{2i} \log (uv|\alpha\alpha') \tag{1}$$

where α and α' are the isotropic lines on the vertex of the angle.

Likewise the measure of distance is elliptic and we write[1]

$$\text{dist } xy = \frac{k}{2i} \log (xy|aa') \tag{2}$$

where a and a' are the absolute points collinear with x and y. The length of a quadrant is now $k\pi/2$. Further if u and v are the polars of x and y, then

$$\text{dist } xy = k \text{ ang } uv. \tag{3}$$

Thus the equations involving distance in hyperbolic geometry are valid in elliptic geometry if only ik is replaced by k. In particular, the equation of the absolute may be written in the form

$$x^2 + y^2 + k^2z^2 = 0. \tag{4}$$

The elliptic plane like the elliptic line is finite but unbounded. It contains a single class of real points, the actual points and a single class of real lines, the actual lines. Again there is only one type of line pairs, the intersectors, *i. e., two lines meet without exception at a finite distance.*

Two lines which are perpendicular to the same line l meet

[1] This disagrees of course with the parabolic measure in Euclidean and the hyperbolic measure in hyperbolic geometry.

in the absolute pole of l. But by symmetry they meet in either direction and we must either say that the lines meet in two points or agree that these two points are identical. The latter convention is accepted for elliptic geometry.[1] All lines of the elliptic plane are thus of the same finite length,—*viz.*, two quadrants $2q$ or $k\pi$. And it is obvious that the locus of a point at a constant distance d from a point is at the same time the locus of a point at a distance $q - d$ from the polar of the point. Hence all non-degenerate circles are at once proper circles and equidistant curves.

The equation of a circle with center (c_1, c_2, c_3) and radius r becomes

$$\cos^2\frac{r}{k} = \frac{A_{cx}{}^2}{A_{cc}A_{xx}} \tag{5}$$

or if the absolute is taken in the form (4), the equation of the circle is

$$\cos^2\frac{r}{k} = \frac{(c_1x + c_2y + c_3k^2z)^2}{(c_1{}^2 + c_2{}^2 + k^2c_3{}^2)(x^2 + y^2 + k^2z^2)}. \tag{6}$$

If (6) is regarded as an equidistant curve with distance d from the axis we have

$$d = k\pi/2 - r \quad \text{or} \quad d/k = \pi/2 - r/k$$

and (6) may be written

$$\sin^2\frac{d}{k} = \frac{(c_1x + c_2y + c_3k^2z)^2}{(c_1{}^2 + c_2{}^2 + k^2c_3{}^2)(x^2 + y^2 + k^2z^2)} \tag{7}$$

where $c_1x + c_2y + c_3k^2z = 0$ is the equation of the axis and (c_1, c_2, c_3) is the pole of the axis. Or if the equation of the axis is $ux + vy + wz = 0$, the pole of the axis (§136) is

[1] When the two points are regarded as distinct we get another species of (elliptic) non-Euclidean geometry which closely resembles the Euclidean geometry on a sphere and is called therefore *spherical* or *antipodal*. The great circle in Euclidean spherical plays the role of the straight line in antipodal geometry. The elliptic geometry of this section is analogous to Euclidean geometry on an unbounded hemisphere.

$(u, v, w/k^2)$. Hence *the equation of an equidistant curve with distance* d and *axis* (u, u, w) is

$$\sin^2 \frac{d}{k} = \frac{(ux + vy + wz)^2}{(u^2 + v^2 + w^2/k^2)(x^2 + y^2 + k^2z^2)}. \tag{8}$$

Now (8) involves the point and line equations of the absolute symmetrically. Hence for variable u, v, w and constant x, y, z it is equally well the line equation of a circle with center (x, y, z) and axis (x, y, k^2z).

188. Non-Euclidean properties derived from projective. As indicated (§180) projective theorems involving a conic give rise to non-Euclidean theorems when the conic is projected into the absolute. Since any two non-degenerate conics are projectively equivalent, any proper conic can be taken for the absolute. Thus Pascal's theorem becomes in non-Euclidean geometry:

If six lines 1, 2, 3, 4, 5, 6 are so related that 1 is parallel to 2, 2 is parallel to 3 and so on in order, 6 being parallel to 1, but such that no three lines are parallel in the same sense,[1] then the pairs 1, 4; 2, 5 and 3, 6 meet in three collinear points.

Indeed certain Euclidean theorems can be translated into non-Euclidean by first changing them into projective form as in §79 and then isolating a conic for the absolute. In particular properties relating to systems of (*a*) concentric circles, (*b*) conics with a common focus and corresponding directrix,[2] (*c*) conics with the same asymptotes go over into non-Euclidean properties of circles. For each of the three systems is projectively equivalent to a pencil of double contact conics. Hence if one member of the system is pro-

[1] Three lines are parallel in the same sense if they meet at the same point at infinity. While line 2 is parallel to both 1 and 3 the three lines are not parallel in the same sense, *i. e.*, line 1 cuts 2 in one of its infinite points and 3 cuts it in the other.

[2] Except parabolas of course.

jected into the absolute conic the other members go into concentric circles. Frequently we get two or more non-Euclidean theorems from a single projective theorem by choosing different conics for the absolute.

Thus the projective statement of Ex. 1, §79 is: (1) If two conics have double contact a tangent to either cuts the other in two points which are harmonically separated by the point of contact and the intersection of the tangent with the common chord.

If now one conic is projected into the absolute the other will become a circle. Then taking the tangent to be first a line of the circle and then of the absolute we obtain the two non-Euclidean theorems:

(1′) A tangent to a circle cuts the absolute in two points which are harmonically separated by the point of contact and the point of intersection of the tangent and the axis of the circle.

(1″) An isotropic line cuts a circle in two points which are harmonically separated by the point of contact of the line and the intersection of the line with the axis of the circle.

EXERCISES

1. Select the exercises of this chapter which are valid in elliptic geometry.

2. The locus of lines which are perpendicular to themselves is the absolute. (All three geometries.)

3. Pairs of perpendicular lines on a point belong to a quadratic involution whose double lines are the isotropic lines on the point. (All three geometries.)

4. Pairs of points on a line which are conjugate with respect to the absolute belong to an involution whose double points are the absolute points of the line. The involution is elliptic or hyperbolic according as the geometry is elliptic or hyperbolic.

5. If A, B, C are the angles of a triangle and a', b', c' are the cor-

responding sides of the polar triangle, show (as in Euclidean spherical geometry, *e. g.*) that

$$a' + b' + c' = \frac{2q}{\pi}(3\pi - A - B - C)$$

where q is a quadrant distance.

Prove the following theorems by the method suggested in §188.

6. The axes of two circles and one pair of common chords meet in a point and form a harmonic pencil. (Ex. 17, §140—take S for the absolute.)

7. In a system of concentric circles, a tangent to one cuts the others in pairs of points in an involution whose double points are the point of contact of the tangent and the intersection of the tangent with the common chord of the system (common axis of the circles), *i. e.*, the chords cut from the system by the tangent are all bisected by the contact and the common axis. (All three geometries.) *Cf.* theorem of Desargues, §90.

8. Given a line l and a point P. Then if the pencil formed by the two isotropic lines on P and the two lines on P parallel to l has a constant double ratio, the locus of P is a circle of which l is the axis. (Example 1, §79.)

9. The envelope of the line joining the contacts of the isotropic lines in Ex. 8 is a circle concentric with the other. (Ex. 7, §79.)

10. Translate into a non-Euclidean theorem: The locus of a point from which the tangents to two conics form a harmonic pencil is a conic on the contacts of the common lines of the two conics. Dualize both the projective and the non-Euclidean statements.

11. Obtain a projective theorem from the non-Euclidean theorem: The locus of a point at a fixed distance from a given point is a circle whose axis is the absolute polar of the point. (Replace the absolute by a general conic.) Dualize.

189. Historical Sketch.—The purpose of this chapter has been to present the fundamental properties of the classical non-Euclidean geometries in their relation to projective geometry as well as to one another. In the early chapters of the book we emphasized the connection between projective geometry and Euclid. We shall be content if we have been able to exhibit to the student a picture of these three metric geometries as specialized forms of projective

geometry, depending for their respective peculiarities on the nature of the locus which is isolated for the absolute. He will then see that while the three geometries are in frequent conflict with one another, each is valid in its own domain and one is as true as another in a mathematical sense. And he will be spared the view,—only too common even today,—that non-Euclidean geometry is a grotesque and monstrous product of disordered imaginations. Indeed he will realize that a knowledge of non-Euclidean geometry is essential to a proper appreciation of Euclid, since Euclid is but a degenerate form of the others.

While the projective approach to non-Euclidean geometry seems at once the most natural and the most elegant it is not so well beaten as other paths. It is to the theory of parallels that we owe the origin and early development of non-Euclidean geometry. An astonishingly large part of Euclid's geometry depends on his fifth (parallel) postulate: If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles. This is equivalent to the modern version: Through a point one and only one line can be drawn parallel to a given line. It is also equivalent to the proposition that the sum of the angles of a triangle is two right angles. And if infinite elements are accepted it amounts to assuming the parabolic type of line.

The parallel postulate was not acceptable to the followers of Euclid and many ingenious but unsuccessful attempts were made to deduce it from the other axioms.[1] These

[1] After such simple and plausible statements as the other postulates: to draw a straight line from any point to any point; to produce a line segment continuously in a straight line; to describe a circle with any center and distance; all right angles are equal to one another; it is not surprising that such an involved proposition as the parallel postulate should have been challenged.

efforts continued for some two thousand years when in a notable attempt to vindicate Euclid[1] Saccheri (1667–1733) proposed two alternative hypotheses, one of which leads to two parallels to a line through a point and to the hyperbolic geometry while the other leads to elliptic geometry in which two lines always meet. He obtained many of the fundamental properties of the elliptic and hyperbolic geometries and then, determined to establish the authority of Euclid, he "proved" his hypotheses false thus failing to attain the rank of founder of non-Euclidean geometry.

Little progress was made until about a hundred years later when Gauss (1777–1855), his friends and pupils became deeply interested in the subject. Gradually the conviction crystallized that Euclid's postulate could not be demonstrated, that indeed a consistent geometry could be developed if it were replaced by a contradictory one. Gauss however did not publish his results and his contributions are found only in letters. It remained for John Bolyai (1802–1860) of Hungary and Lobachevsky (1793–1856) of Russia to announce their independent and almost simultaneous discovery of the hyperbolic geometry. The former when only twenty-one years old, in youthful ecstacy communicated his wonderful discovery to his father Wolfgang Bolyai, a professor of mathematics who had himself long struggled with the problem of parallels. His results were included in a volume published by his father in 1832.[2] Lobachevsky was a professor of mathematics at the University of Kazan who concerned himself with the theory of parallels at an early age. In a paper read at the university

[1] *Euclides ab omni naevo vindicatus*, Milan, 1733. English by Halsted, *American Mathematical Monthly*, Vols. 1–5 and Chicago, 1920.

[2] Under the title: "*Appendix, Scientiam Absolute Veram Exhibens.*" This Appendix which constitutes John Bolyai's sole publication Halsted characterizes as "the most extraordinary two dozen pages in the history of thought." Translated into English by Halsted, Austin, Texas, 1891.

in 1826 he communicated his discovery of hyperbolic geometry which he called "imaginary geometry." He embodied his results in a book published in 1840.[1] Since Lobachevsky's developments are more complete than those of Bolyai, the hyperbolic geometry is frequently referred to under his name.

Curiously enough the possibility of a consistent geometry without (real) parallel lines was unsuspected till it was implied by Riemann (1826–1866) in his memorable dissertation of 1854.[2] Riemann's geometry was however of the spherical or antipodal type, §187, footnote. The conception of elliptic geometry in which two lines meet in a single point is due to Klein.[3]

The denial of Euclid's postulate carries with it of course the denial of the proposition that the sum of the angles of a triangle is two right angles. In hyperbolic geometry this sum is always less than two right angles while in elliptic (as in Euclidean spherical) it is always greater. In either case the deficiency or the excess is proportional to the area of the triangle. Similar figures therefore do not exist in non-Euclidean geometry.

Doubts concerning the consistency of the new geometries were allayed finally by Beltrami (1835–1900) who exhibited a representation of the hyperbolic geometry in Euclidean space,[4] thus identifying the problem of the consistency of non-Euclidean geometry with that of the consistency of Euclid itself—and no one questions the consistency of Euclid.

[1] *Geometrische Untersuchungen zur Theorie der Parallellinien*, translated by Halsted, Austin, Texas, 1891.

[2] *Über die Hypothesen welche der Geometrie zu Grunde liegen*, English by Clifford, Collected Papers.

[3] *Über die sogenannten Nicht-Euklidische Geometrie*, *Math. Annalen*, Vol. 4, 1871.

[4] *Saggio di interpretazione della geometria non-euclidea*, *Giornale di Math.* Vol. 6, 1868. Likewise the geometry on the *horosphere*, a non-Euclidean sphere of infinite radius, is Euclidean.

Cayley[1] introduced the notion of the absolute and showed the connection between the non-Euclidean systems and projective geometry. Klein[2] supplemented Cayley's work and supplied the logarithmic expressions for distance and angle. He also suggested the names elliptic, hyperbolic and parabolic for the three geometries. In this chapter we have chiefly followed Klein's exposition.

The history of non-Euclidean geometry is one of the most interesting as well as one of the most instructive chapters in the history of mathematics. One lesson to be emphasized is that a great scientific creation is rarely the work of one man, even though he be a genius, but more often represents the composite labors of many minds. Non-Euclidean geometry may be said to have evolved through a period of over 2,000 years until several men in various countries caught the great vision that the geometry of Euclid, flawless as it is in its own domain, is not the only true system of geometry. Not the least important product, or perhaps we should say by-product, of the long research was that men were led to examine anew the basis of geometry and the foundations of mathematics. And mathematicians came as a consequence to understand for the first time the true nature of their science. Finally the insight into the structure of mathematics has pointed the way to reforms in teaching which are today yielding bountiful fruit.

The existence of non-Euclidean geometry proves once for all the impossibility of deriving the parallel postulate from the other axioms, that the fifth was as indispensable as any other postulate to the development of Euclid's system. It is therefore a monument to the genius of the great geometer that he placed the controversial proposition among the postulates and Saccheri's dream is realized at

[1] A sixth Memoir upon Quantics, Phil. Trans. R. Soc., 1859.
[2] *L.c.* and in his Lectures cited below.

last,—but how different from the manner he anticipated!—*Euclides ab omni naevo vindicatus.*

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